

# Spectral gap for stable process on convex planar double symmetric domains

Bartłomiej Dyda and Tadeusz Kulczycki

Institute of Mathematics and Computer Science

Wrocław University of Technology

Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

February 2, 2008

## Abstract

We study the semigroup of the symmetric  $\alpha$ -stable process in bounded domains in  $\mathbf{R}^2$ . We obtain a variational formula for the spectral gap, i.e. the difference between two first eigenvalues of the generator of this semigroup. This variational formula allows us to obtain lower bound estimates of the spectral gap for convex planar domains which are symmetric with respect to both coordinate axes. For rectangles, using "midconcavity" of the first eigenfunction [5], we obtain sharp upper and lower bound estimates of the spectral gap.

## 1 Introduction

In recent years many results have been obtained in spectral theory of semigroups of symmetric  $\alpha$ -stable processes  $\alpha \in (0, 2)$  in bounded domains in  $\mathbf{R}^d$ , see [6], [25], [2], [18], [19], [14], [15], [5]. One of the most interesting problems in spectral theory of such semigroups is a spectral gap estimate i.e. the estimate of  $\lambda_2 - \lambda_1$  the difference between two first eigenvalues of the generator of this semigroup. Such estimate is a natural generalisation of the same problem for the semigroup of Brownian motion killed on exiting

---

<sup>0</sup>*Key words and phrases:* symmetric stable process, spectral gap, convex domain

The first named author was supported by KBN grant 1 P03A 026 29 and RTN Harmonic Analysis and Related Problems, contract HPRN-CT-2001-00273-HARP

The second named author was supported by KBN grant 1 P03A 020 28 and RTN Harmonic Analysis and Related Problems, contract HPRN-CT-2001-00273-HARP.

a bounded domain, which generator is Dirichlet Laplacian. In this classical case, for Brownian motion, spectral gap estimates have been widely studied see e.g [26], [28], [24], [27], [17], [7]. When a bounded domain is convex there have been obtained sharp lower-bound estimates of the spectral gap.

In the case of the semigroup of symmetric  $\alpha$ -stable processes  $\alpha \in (0, 2)$  very little is known about the spectral gap estimates. In one dimensional case when a domain is just an interval spectral gap estimates follow from results from [2] ( $\alpha = 1$ ) and [14] ( $\alpha > 1$ ). The only results for dimension greater than one have been obtained for the Cauchy process i.e.  $\alpha = 1$  [3], [4]. Such results have been obtained using the deep connection between the eigenvalue problem for the Cauchy process and a boundary value problem for the Laplacian in one dimension higher, known as the mixed Steklov problem.

The aim of this paper is to generalise spectral gap estimates obtained for the Cauchy process ( $\alpha = 1$ ) for all  $\alpha \in (0, 2)$ . Before we describe our results in more detail let us recall definitions and basic facts.

Let  $X_t$  be a symmetric  $\alpha$ -stable process in  $\mathbf{R}^d$ ,  $\alpha \in (0, 2]$ . This is a process with independent and stationary increments and characteristic function  $E^0 e^{i\xi X_t} = e^{-t|\xi|^\alpha}$ ,  $\xi \in \mathbf{R}^d$ ,  $t > 0$ . We will use  $E^x$ ,  $P^x$  to denote the expectation and probability of this process starting at  $x$ , respectively. By  $p(t, x, y) = p_t(x - y)$  we will denote the transition density of this process. That is,

$$P^x(X_t \in B) = \int_B p(t, x, y) dy.$$

When  $\alpha = 2$  the process  $X_t$  is just the Brownian motion in  $\mathbf{R}^d$  running at twice the speed. That is, if  $\alpha = 2$  then

$$p(t, x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}, \quad t > 0, \quad x, y \in \mathbf{R}^d. \quad (1.1)$$

It is well known that for  $\alpha \in (0, 2)$  we have  $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x)$ ,  $t > 0$ ,  $x \in \mathbf{R}^d$  and

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x) \leq t^{-d/\alpha} p_1(0) = t^{-d/\alpha} M_{d,\alpha}, \quad t > 0, \quad x \in \mathbf{R}^d,$$

where

$$M_{d,\alpha} = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-|x|^\alpha} dx. \quad (1.2)$$

It is also well known that

$$\lim_{t \rightarrow 0^+} \frac{p(t, x, y)}{t} = \frac{\mathcal{A}_{d,-\alpha}}{|x - y|^{d+\alpha}}, \quad (1.3)$$

where

$$\mathcal{A}_{d,\gamma} = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|). \quad (1.4)$$

Our main concern in this paper are the eigenvalues of the semigroup of the process  $X_t$  killed upon leaving a domain. Let  $D \subset \mathbf{R}^d$  be a bounded connected domain and  $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$  be the first exit time of  $D$ . By  $\{P_t^D\}_{t \geq 0}$  we denote the semigroup on  $L^2(D)$  of  $X_t$  killed upon exiting  $D$ . That is,

$$P_t^D f(x) = E^x(f(X_t), \tau_D > t), \quad x \in D, \quad t > 0, \quad f \in L^2(D).$$

The semigroup has transition densities  $p_D(t, x, y)$  satisfying

$$P_t^D f(x) = \int_D p_D(t, x, y) f(y) dy.$$

The kernel  $p_D(t, x, y)$  is strictly positive symmetric and

$$p_D(t, x, y) \leq p(t, x, y) \leq M_{d,\alpha} t^{-d/\alpha}, \quad x, y \in D, \quad t > 0.$$

The fact that  $D$  is bounded implies that for any  $t > 0$  the operator  $P_t^D$  maps  $L^2(D)$  into  $L^\infty(D)$ . From the general theory of semigroups (see [16]) it follows that there exists an orthonormal basis of eigenfunctions  $\{\varphi_n\}_{n=1}^\infty$  for  $L^2(D)$  and corresponding eigenvalues  $\{\lambda_n\}_{n=1}^\infty$  satisfying

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

with  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . That is, the pair  $\{\varphi_n, \lambda_n\}$  satisfies

$$P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x), \quad x \in D, \quad t > 0. \quad (1.5)$$

The eigenfunctions  $\varphi_n$  are continuous and bounded on  $D$ . In addition,  $\lambda_1$  is simple and the corresponding eigenfunction  $\varphi_1$ , often called the ground state eigenfunction, is strictly positive on  $D$ . By scaling we have for  $\beta > 0$

$$\lambda_n(\beta D) = \beta^{-\alpha} \lambda_n(D). \quad (1.6)$$

For more general properties of the semigroups  $\{P_t^D\}_{t \geq 0}$ , see [21], [8], [12].

It is well known (see [1], [12], [13], [23]) that if  $D$  is a bounded connected Lipschitz domain and  $\alpha = 2$ , or that if  $D$  is a bounded connected domain for  $0 < \alpha < 2$ , then  $\{P_t^D\}_{t \geq 0}$  is intrinsically ultracontractive. Intrinsic ultracontractivity is a remarkable property with many consequences. It implies, in particular, that

$$\lim_{t \rightarrow \infty} \frac{e^{\lambda_1 t} p_D(t, x, y)}{\varphi_1(x) \varphi_1(y)} = 1,$$

uniformly in both variables  $x, y \in D$ . In addition, the rate of convergence is given by the spectral gap  $\lambda_2 - \lambda_1$ . That is, for any  $t \geq 1$  we have

$$e^{-(\lambda_2 - \lambda_1)t} \leq \sup_{x, y \in D} \left| \frac{e^{\lambda_1 t} p_D(t, x, y)}{\varphi_1(x) \varphi_1(y)} - 1 \right| \leq C(D, \alpha) e^{-(\lambda_2 - \lambda_1)t}. \quad (1.7)$$

The proof of this for  $\alpha = 2$  may be found in [27]. The proof in our setting is exactly the same.

Our first step in studying the spectral gap for  $\alpha \in (0, 2)$  is the following variational characterisation of  $\lambda_2 - \lambda_1$ .

By  $L^2(D, \varphi_1^2)$  we denote the  $L^2$  space of functions with the inner product  $(f, g)_{L^2(D, \varphi_1^2)} = \int_D f(x) g(x) \varphi_1^2(x) dx$ .

**Theorem 1.1.** *We have*

$$\lambda_2 - \lambda_1 = \inf_{f \in \mathcal{F}} \frac{\mathcal{A}_{d, -\alpha}}{2} \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \varphi_1(x) \varphi_1(y) dx dy, \quad (1.8)$$

where

$$\mathcal{F} = \{f \in L^2(D, \varphi_1^2) : \int_D f^2(x) \varphi_1^2(x) dx = 1, \int_D f(x) \varphi_1^2(x) dx = 0\}$$

and  $\mathcal{A}_{d, -\alpha}$  is given by (1.4). Moreover the infimum is achieved for  $f = \varphi_2 / \varphi_1$ .

The idea of the proof is based on considering a new semigroup  $\{T_t^D\}_{t \geq 0}$  of the stable process conditioned to remain forever in  $D$ . The proof of Theorem 1.1 is in Section 2.

In the classical case, for Brownian motion, when a dimension is greater than one, the simplest domain where the spectral gap can be explicitly calculated is a rectangle. Let us recall that in this classical case  $\{\varphi_n\}_{n=1}^\infty$ ,  $\{\lambda_n\}_{n=1}^\infty$  are of course eigenfunctions and eigenvalues of Dirichlet Laplacian. Therefore, when (say)  $D = (-a, a) \times (-b, b)$ ,  $a \geq b > 0$  then

$$\varphi_1(x_1, x_2) = (1/\sqrt{2ab}) \cos(\pi x_1/(2a)) \cos(\pi x_2/(2b)),$$

$$\varphi_2(x_1, x_2) = (1/\sqrt{2ab}) \sin(2\pi x_1/(2a)) \cos(\pi x_2/(2b)),$$

$\lambda_1 = \pi^2/(4a^2) + \pi^2/(4b^2)$ ,  $\lambda_2 = 4\pi^2/(4a^2) + \pi^2/(4b^2)$  and hence  $\lambda_2 - \lambda_1 = 3\pi^2/(4a^2)$ .

Although the  $\alpha$ -stable process is generated by  $-(-\Delta)^{\alpha/2}$ , the generator of the killed  $\alpha$ -stable process on  $D$  is however not equal to  $-(-\Delta_D)^{\alpha/2}$  for the Dirichlet Laplacian  $\Delta_D$  on  $D$ . So, both  $\varphi_n$  and  $\lambda_n$  are not explicit even for an interval or a rectangle. However, when  $D$  is a rectangle, due

to simple geometric properties of this set it is shown ([5] Theorem 1.1) that the first eigenfunction  $\varphi_1$  for any  $\alpha \in (0, 2]$  is "midconcave" and unimodal according to the lines parallel to the sides. This property and Theorem 1.1 enables us to obtain sharp upper and lower bound estimates of the spectral gap for all  $\alpha \in (0, 2)$ . The most complicated are lower bound estimates for  $\alpha \in (1, 2)$  and  $\alpha = 1$ . The main idea of the proof in these cases is contained in Lemmas 4.2 and 4.3.

Below we present estimates of  $\lambda_2 - \lambda_1$  for rectangles. The proof of this theorem is in Section 4. Let us point out that these estimates are sharp i.e. the upper and lower bound estimates have the same dependence on the length of the sides of the rectangle. Nevertheless, the numerical constants which appear in this theorem are far from being optimal.

**Theorem 1.2.** *Let  $D = (-a, a) \times (-b, b)$ , where  $a \geq b$ . Then*

(a) *We have*

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \leq 10^6 \cdot \begin{cases} \frac{2}{1-\alpha} \frac{b}{a^{1+\alpha}} & \text{for } \alpha < 1, \\ 2 \log\left(1 + \frac{a}{b}\right) \frac{b}{a^2} & \text{for } \alpha = 1, \\ \left(\frac{1}{2-\alpha} + \frac{1}{\alpha-1}\right) \frac{b^{2-\alpha}}{a^2} & \text{for } \alpha > 1. \end{cases}$$

(b) *We have*

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \geq \begin{cases} \frac{b}{36 \cdot 2^{1+2\alpha} a^{1+\alpha}} & \text{for } \alpha < 1, \\ 10^{-9} \log\left(1 + \frac{a}{b}\right) \frac{b}{a^2} & \text{for } \alpha = 1, \\ \frac{1}{33 \cdot 13^{1+\alpha/2} \cdot 10^4} \frac{b^{2-\alpha}}{a^2} & \text{for } \alpha > 1. \end{cases}$$

Let us note that for  $\alpha = 1$  the following estimates have already been known  $\lambda_2 - \lambda_1 \geq Cb/a^2$ , where  $C = 10^{-7}$  (Corollary 1.1, [4]). However, estimates from Theorem 1.2 are more precise because we get an extra term  $\log(a/b + 1)$ , which gives a sharp dependence on the length of the sides of a rectangle.

**Remark 1.3.** *The inequality*

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \geq \frac{b}{36 \cdot 2^\alpha (a+b)^{1+\alpha}}$$

holds for all  $\alpha \in (0, 2)$ .

We have  $2\mathcal{A}_{2,-\alpha}^{-1} = \alpha^{-2}2^{3-\alpha}\pi\Gamma^{-1}(\alpha/2)\Gamma(1-\alpha/2)$ . In particular we get for example  $\lambda_2 - \lambda_1 \geq \frac{8b}{10^4(a+b)^{3/2}}$  for  $\alpha = 1/2$ ,  $\lambda_2 - \lambda_1 \geq \frac{b}{10^3(a+b)^2}$  for  $\alpha = 1$ ,  $\lambda_2 - \lambda_1 \geq \frac{8b}{10^4(a+b)^{5/2}}$  for  $\alpha = 3/2$ .

Our next aim are lower bound estimates of the spectral gap for convex planar domains which are symmetric with respect to both coordinate axes.

In the classical case, for the Brownian motion, there are known sharp estimates for all bounded convex domains  $D \subset \mathbf{R}^d$ . We have  $\lambda_2 - \lambda_1 > \pi^2/d_D^2$  where  $d_D$  is the diameter of  $D$  see e.g. [24], [27]. Such results are obtained using the fact that the first eigenfunction is log-concave. For convex planar domains which are symmetric with respect to both coordinate axes even better estimates  $\lambda_2 - \lambda_1 > 3\pi^2/d_D^2$  are known, see [17], [7] (such estimates are optimal, the lower bound is approached by this rectangles). These results follow from ratio inequalities for heat kernels.

Unfortunately in the case of symmetric  $\alpha$ -stable processes,  $\alpha \in (0, 2)$ , we do not know whether the first eigenfunction is log-concave. Instead we use some of the ideas from [4] where spectral gap estimates for the Cauchy process i.e.  $\alpha = 1$  were obtained. Namely, we use the fact that the first eigenfunction is unimodal according to the lines parallel to coordinate axes and that it satisfies the appropriate Harnack inequality. Then we use similar techniques as for rectangles. As before in this proof the crucial role have Lemmas 4.2 and 4.3.

The properties of the first eigenfunction are obtained in Section 3 and the proof of lower bound estimates for the spectral gap is in Section 5. These estimates we present below in Theorem 1.4. Let us point out that these estimates are sharp only for  $\alpha > 1$ , where we know that they cannot be improved because of the results for rectangles.

**Theorem 1.4.** *Let  $D \subset \mathbf{R}^2$  be a bounded convex domain which is symmetric relative to both coordinate axes. Assume that  $[-a, a] \times [-b, b]$ ,  $a \geq b$  is the smallest rectangle (with sides parallel to the coordinate axes) containing  $D$ . Then we have*

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \geq \frac{C b^{2-\alpha}}{a^2},$$

where

$$C = C(\alpha) = 10^{-9}3^{\alpha-4}2^{-2\alpha-1} \left( 4 + \frac{12\Gamma(2/\alpha)}{\alpha(2-\alpha)(1-2^{-\alpha})^{2/\alpha}} \right)^{-2}. \quad (1.9)$$

Let us note that for  $\alpha = 1$  such estimate has already been known with a better constant. In fact, Corollary 1.1 [4] gives  $\lambda_2 - \lambda_1 \geq Cb/a^2$ , where  $C = 10^{-7}$ .

There are still many open problems concerning the spectral gap for semi-groups of symmetric stable processes  $\alpha \in (0, 2)$  in bounded domains  $D \subset \mathbf{R}^d$ . Perhaps the most interesting is the following. What is the best possible lower bound estimate for the spectral gap for arbitrary bounded convex domain  $D \subset \mathbf{R}^d$ ? With this problem there are connected questions about the shape of the first eigenfunction  $\varphi_1$ . For example, is  $\varphi_1$  log-concave or at least unimodal when  $D$  is a convex bounded domain? There is also an unsolved problem concerning domains from Theorem 1.4. Can one obtain for  $\alpha \leq 1$  lower bounds similar to these obtained for rectangles i.e.  $\lambda_2 - \lambda_1 \geq C_\alpha b/a^{1+\alpha}$  for  $\alpha < 1$  and  $\lambda_2 - \lambda_1 \geq C b \log(1 + a/b)/a^2$  for  $\alpha = 1$ ?

## 2 Variational formula

In this section we prove Theorem 1.1 – the variational formula for the spectral gap.

At first we need the following simple properties of the kernel  $p_D(t, x, y)$ .

**Lemma 2.1.** *There exists a constant  $c = c(d, \alpha)$  such that for any  $t > 0$ ,  $x, y \in D$  we have*

$$p_D(t, x, y) \leq p(t, x, y) \leq \frac{ct}{|x - y|^{d+\alpha}}. \quad (2.1)$$

For any  $x, y \in D$ ,  $x \neq y$  we have

$$\lim_{t \rightarrow 0^+} \frac{p_D(t, x, y)}{t} = \lim_{t \rightarrow 0^+} \frac{p(t, x, y)}{t} = \frac{\mathcal{A}_{d, -\alpha}}{|x - y|^{d+\alpha}}. \quad (2.2)$$

*Proof.* These properties of  $p_D(t, x, y)$  are rather well known. We recall some of the standard arguments.

The estimate  $p(t, x, y) \leq ct|x - y|^{-d-\alpha}$  follows e.g. from the scaling property  $p(t, x, y) = t^{-d/\alpha} p_1((x - y)t^{1/\alpha})$  and the inequality  $p_1(z) \leq c|z|^{-d-\alpha}$  [29]. The equality on the right-hand side of (2.2) is well known (see (1.3)).

We know that  $p_D(t, x, y) = p(t, x, y) - r_D(t, x, y)$  where

$$r_D(t, x, y) = E^x(\tau_D < t; p(t - \tau_D, X(\tau_D), y)).$$

By (2.1) we get for  $x, y \in D$ ,  $t > 0$

$$\begin{aligned} \frac{1}{t} r_D(t, x, y) &= \frac{1}{t} E^x(\tau_D < t; p(t - \tau_D, X(\tau_D), y)) \\ &\leq \frac{1}{t} E^x \left( \tau_D < t; \frac{ct}{|y - X(\tau_D)|^{d+\alpha}} \right) \\ &\leq \frac{cP^x(\tau_D < t)}{(\delta_D(y))^{d+\alpha}}, \end{aligned}$$

where  $\delta_D(y) = \inf\{|z - y| : z \in \partial D\}$ . It follows that  $t^{-1}r_D(t, x, y) \rightarrow 0$  when  $t \rightarrow 0^+$ .  $\square$

Let

$$\tilde{p}_D(t, x, y) = \frac{e^{\lambda_1 t} p_D(t, x, y)}{\varphi_1(x) \varphi_1(y)}, \quad x, y \in D, \quad t > 0$$

and

$$T_t^D f(x) = \int_D \tilde{p}_D(t, x, y) f(y) \varphi_1^2(y) dy, \quad f \in L^2(D, \varphi_1^2), \quad t > 0.$$

$\{T_t^D\}_{t \geq 0}$  is a semigroup in  $L^2(D, \varphi_1^2)$ . This is the semigroup for the stable process conditioned to remain forever in  $D$  (see [27] where the same semigroup is defined for Brownian motion).

Let

$$\mathcal{E}(f, f) = \lim_{t \rightarrow 0^+} \frac{1}{t} (f - T_t^D f, f)_{L^2(D, \varphi_1^2)},$$

for  $f \in L^2(D, \varphi_1^2)$ .

**Lemma 2.2.** *For any  $f \in L^2(D, \varphi_1^2)$   $\mathcal{E}(f, f)$  is well defined and we have*

$$\mathcal{E}(f, f) = \frac{\mathcal{A}_{d, -\alpha}}{2} \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \varphi_1(x) \varphi_1(y) dx dy. \quad (2.3)$$

*Proof.*

$$\begin{aligned} \mathcal{E}(f, f) &= \lim_{t \rightarrow 0^+} \frac{1}{t} (f - T_t^D f, f)_{L^2(D, \varphi_1^2)} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t} \int_D \left( f(x) - \int_D \frac{e^{\lambda_1 t} p_D(t, x, y)}{\varphi_1(x) \varphi_1(y)} f(y) \varphi_1^2(y) dy \right) f(x) \varphi_1^2(x) dx \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t} \int_D \left( f(x) \varphi_1(x) - e^{\lambda_1 t} \int_D p_D(t, x, y) f(y) \varphi_1(y) dy \right) \\ &\quad \times f(x) \varphi_1(x) dx. \end{aligned} \quad (2.4)$$

Note that

$$f(x) \varphi_1(x) = f(x) e^{\lambda_1 t} P_t^D \varphi_1(x) = e^{\lambda_1 t} \int_D p_D(t, x, y) f(x) \varphi_1(y) dy.$$

Hence (2.4) is equal to

$$\begin{aligned} &\lim_{t \rightarrow 0^+} \frac{1}{t} \int_D e^{\lambda_1 t} \int_D p_D(t, x, y) (f(x) \varphi_1(y) - f(y) \varphi_1(y)) dy f(x) \varphi_1(x) dx \\ &= \lim_{t \rightarrow 0^+} e^{\lambda_1 t} \int_D \int_D \frac{p_D(t, x, y)}{t} (f^2(x) - f(x) f(y)) \varphi_1(x) \varphi_1(y) dy dx. \end{aligned} \quad (2.5)$$

Note that we can interchange the role of  $x$  and  $y$  in (2.5). Therefore by standard arguments (2.5) is equal to

$$\lim_{t \rightarrow 0^+} \frac{e^{\lambda_1 t}}{2} \int_D \int_D \frac{p_D(t, x, y)}{t} (f(x) - f(y))^2 \varphi_1(x) \varphi_1(y) dx dy. \quad (2.6)$$

In view of (2.2) in order to prove (2.3) we need only to justify the interchange of the limit and the integral in (2.6). Let us denote

$$\mathcal{E}_1(f, f) = \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \varphi_1(x) \varphi_1(y) dx dy.$$

When  $\mathcal{E}_1(f, f) = \infty$  then (2.3) follows from (2.6) by the Fatou lemma. Now let us consider the case  $\mathcal{E}_1(f, f) < \infty$ . By (2.1) for any  $t > 0$  we have

$$\frac{p_D(t, x, y)}{t} (f(x) - f(y))^2 \varphi_1(x) \varphi_1(y) \leq \frac{c(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \varphi_1(x) \varphi_1(y). \quad (2.7)$$

The integral over  $D \times D$  of the right-hand side of (2.7) is equal to  $c\mathcal{E}_1(f, f) < \infty$ . Now (2.3) follows from (2.6) by the bounded convergence theorem.  $\square$

*Proof of Theorem 1.1.* Let  $f \in \mathcal{F}$ . We have  $f\varphi_1 \in L^2(D)$ ,  $\|f\varphi_1\|_{L^2(D)} = 1$  and  $f\varphi_1 \perp \varphi_1$  in  $L^2(D)$ . Since  $\{\varphi_n\}_{n=1}^\infty$  is an orthonormal basis in  $L^2(D)$  we have

$$f\varphi_1 = \sum_{n=2}^\infty c_n \varphi_n,$$

where  $c_n = \int_D f(x) \varphi_1(x) \varphi_n(x) dx$  and the equality holds in  $L^2(D)$  sense. Hence

$$f = \sum_{n=2}^\infty c_n \frac{\varphi_n}{\varphi_1}$$

in  $L^2(D, \varphi_1^2)$  sense. The condition  $\|f\varphi_1\|_{L^2(D)} = 1$  gives  $\sum_{n=1}^\infty c_n^2 = 1$ .

We will show that

$$\mathcal{E}(f, f) = \sum_{n=2}^\infty (\lambda_n - \lambda_1) c_n^2. \quad (2.8)$$

We have

$$T_t^D \frac{\varphi_n}{\varphi_1}(x) = \int_D \frac{e^{\lambda_1 t} p_D(t, x, y)}{\varphi_1(x) \varphi_1(y)} \frac{\varphi_n(y)}{\varphi_1(y)} \varphi_1^2(y) dy = e^{-(\lambda_n - \lambda_1)t} \frac{\varphi_n(x)}{\varphi_1(x)}.$$

Hence by Parseval formula

$$(T_t^D f, f)_{L^2(D, \varphi_1^2)} = \sum_{n=2}^{\infty} c_n^2 e^{-(\lambda_n - \lambda_1)t},$$

so

$$\mathcal{E}(f, f) = \lim_{t \rightarrow 0^+} (f - T_t^D f, f)_{L^2(D, \varphi_1^2)} = \lim_{t \rightarrow 0^+} \sum_{n=2}^{\infty} c_n^2 \frac{1 - e^{-(\lambda_n - \lambda_1)t}}{t}. \quad (2.9)$$

To show (2.8) we have to justify the change of the limit and the sum in (2.9). Note that  $(1 - e^{-(\lambda_n - \lambda_1)t})/t \uparrow \lambda_n - \lambda_1$  when  $t \downarrow 0$  by convexity of the exponential function. Hence (2.8) follows from (2.9) by the monotone convergence theorem.

By (2.8) we get

$$\mathcal{E}(f, f) = \sum_{n=2}^{\infty} (\lambda_n - \lambda_1) c_n^2 \geq (\lambda_2 - \lambda_1) \sum_{n=2}^{\infty} c_n^2 = \lambda_2 - \lambda_1.$$

Now Lemma 2.2 shows that the infimum in (1.8) is bigger or equal to  $\lambda_2 - \lambda_1$ . When we put  $f = \varphi_2/\varphi_1$  ( $c_2 = 1$ ,  $c_n = 0$  for  $n \geq 3$ ) we obtain  $\mathcal{E}(\varphi_2/\varphi_1, \varphi_2/\varphi_1) = \lambda_2 - \lambda_1$ . This shows that the infimum in (1.8) is equal to  $\lambda_2 - \lambda_1$  and is achieved for  $f = \varphi_2/\varphi_1$ .  $\square$

### 3 Geometric and Analytic Properties of $\varphi_1$

At first we recall the result which is already proven in [4], Theorem 2.1. (Theorem 2.1 in [4] was formulated for  $\alpha = 1$  (the Cauchy process) but the proof works for all  $\alpha \in (0, 2]$ .)

**Theorem 3.1.** *Let  $D \subset \mathbf{R}^2$  be a bounded convex domain which is symmetric relative to both coordinate axes. Then we have*

- (i)  $\varphi_1$  is continuous and strictly positive in  $D$ .
- (ii)  $\varphi_1$  is symmetric in  $D$  with respect to both coordinate axes. That is,  $\varphi_1(x_1, -x_2) = \varphi_1(x_1, x_2)$  and  $\varphi_1(-x_1, x_2) = \varphi_1(x_1, x_2)$ .
- (iii)  $\varphi_1$  is unimodal in  $D$  with respect to both coordinate axes. That is, if we take any  $a_2 \in (-1, 1)$  and  $p(a_2) > 0$  such that  $(p(a_2), a_2) \in \partial D$ , then the function  $v(x_1) = \varphi_1(x_1, a_2)$  defined on  $(-p(a_2), p(a_2))$  is non-decreasing on  $(-p(a_2), 0)$  and non-increasing on  $(0, p(a_2))$ . Similarly, if we take any  $a_1 \in (-L, L)$  and  $r(a_1) > 0$  such that  $(a_1, r(a_1)) \in \partial D$ , then the function  $u(x_2) = \varphi_1(a_1, x_2)$  defined on  $(-r(a_1), r(a_1))$  is non-decreasing on  $(-r(a_1), 0)$  and non-increasing on  $(0, r(a_1))$ .

Next, we prove the Harnack inequality for  $\varphi_1$ . Such inequality is well known (see e.g. Theorem 6.1 in [10]). Our purpose here is to give a proof which will give an explicit constant. We adopt the method from [4].

At first we need to recall some standard facts concerning stable processes.

By  $P_{r,x}(z, y)$  we denote the Poisson kernel for the ball  $B(x, r) \subset \mathbf{R}^d$ ,  $r > 0$  for the stable process. That is,

$$P^z(X(\tau_{B(x,r)}) \in A) = \int_A P_{r,x}(z, y) dy,$$

where  $z \in B(x, r)$ ,  $A \subset B^c(x, r)$ . We have [9]

$$P_{r,x}(z, y) = C_\alpha^d \frac{(r^2 - |z - x|^2)^{\alpha/2}}{(|y - x|^2 - r^2)^{\alpha/2} |y - z|^d}, \quad (3.1)$$

where  $C_\alpha^d = \Gamma(d/2) \pi^{-d/2-1} \sin(\pi\alpha/2)$ ,  $z \in B(x, r)$  and  $y \in \text{int}(B^c(x, r))$ .

It is well known ([20] cf. also [11] formula (2.10)) that

$$E^y(\tau_{B(0,r)}) = C_\alpha^d (\mathcal{A}_{d,-\alpha})^{-1} (r^2 - |y|^2)^{\alpha/2}, \quad (3.2)$$

where  $r > 0$  and  $\mathcal{A}_{d,-\alpha}$  is given by (1.4).

When  $d > \alpha$  by  $G_D(x, y) = \int_0^\infty p_D(t, x, y) dt$  we denote the Green function for the domain  $D \subset \mathbf{R}^d$ ,  $x, y \in D$ . We have  $G_D(x, y) < \infty$  for  $x \neq y$ . (For  $d \leq \alpha$  the Green function may be defined by a different formula but we will not use it in this paper).

It is well known (see [9]) that

$$G_{B(0,1)}(z, y) = \frac{R_{d,\alpha}}{|z - y|^{d-\alpha}} \int_0^{w(z,y)} \frac{r^{\alpha/2-1} dr}{(r+1)^{d/2}}, \quad z, y \in B(0, 1), \quad (3.3)$$

where

$$w(z, y) = (1 - |z|^2)(1 - |y|^2)/|z - y|^2$$

and  $R_{d,\alpha} = \Gamma(d/2)/(2^\alpha \pi^{d/2} (\Gamma(\alpha/2))^2)$ .

By  $\lambda_1(B_1)$  we denote the first eigenvalue for the unit ball  $B(0, 1)$ . Theorem 4 in [6] (cf. also [14]) gives the following estimate of  $\lambda_1(B_1)$

$$\lambda_1(B_1) \leq (\mu_1(B_1))^{\alpha/2}, \quad (3.4)$$

where  $\mu_1(B_1) \simeq 5.784$  is the first eigenvalue of the Dirichlet Laplacian for the unit ball.

We will also need the following easy scaling property of  $\varphi_1$ .

**Lemma 3.2.** *Let  $D \subset \mathbf{R}^d$  be a bounded domain,  $s > 0$  and  $\varphi_{1,s}$  the first eigenfunction on the set  $sD$  for the stable semigroup  $\{P_t^{sD}\}_{t \geq 0}$ . Then for any  $x \in D$  we have  $\varphi_{1,s}(sx) = s^{-d/2} \varphi_{1,1}(x)$ .*

Now we can formulate the Harnack inequality for  $\varphi_1$ .

**Theorem 3.3.** *Let  $\alpha \in (0, 2)$ ,  $d > \alpha$  and  $D \subset \mathbf{R}^d$  be a bounded domain with inradius  $R > 0$  and  $0 < a < b < 1$ . If  $B(x, bR) \subset D$  then on  $B(x, aR)$   $\varphi_1$  satisfies the Harnack inequality with constant  $C_1 = C_1(d, \alpha, a, b)$ . That is, for any  $z_1, z_2 \in B(x, aR)$  we have  $\varphi_1(z_1) \leq C_1 \varphi_1(z_2)$  where*

$$C_1 = \frac{(b+a)^{d-\alpha/2} b^\alpha}{(b-a)^{d+\alpha/2}} \left( 1 + e + \frac{b^{d+\alpha/2} C_2}{(b-a)^{\alpha/2} (1-b^\alpha)^{d/\alpha}} \right)$$

and  $C_2 = C_2(d, \alpha) = \alpha^2 2^{3d/2-\alpha/2-1} C_\alpha^d M_{d,\alpha}(\lambda_1(B_1))^{d/\alpha} / ((d-\alpha) R_{d,\alpha} \mathcal{A}_{d,-\alpha})$ .

*Proof of Theorem 3.3.* In view of Lemma 3.2 we may and do assume that  $R = 1$ .

Let  $B \subset D$  be any ball ( $B \neq D$ ). For any  $x, y \in B$ ,  $t > 0$  we have

$$p_B(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n(B)t} \varphi_{n,B}(x) \varphi_{n,B}(y), \quad (3.5)$$

where  $\lambda_n(B)$  and  $\varphi_{n,B}$  are the eigenvalues and eigenfunctions for the semigroup  $\{P_t^B\}_{t \geq 0}$ .

We will use the fact that the first eigenfunction is  $q$ -harmonic in  $B$  according to the  $\alpha$ -stable Schrödinger operator.

Let  $\varphi_1$ ,  $\lambda_1 = \lambda_1(D)$  be the first eigenfunction and eigenvalue for the semigroup  $\{P_t^D\}_{t \geq 0}$ . Let  $A$  be the infinitesimal generator of this semigroup. For  $x \in D$  we have

$$A\varphi_1(x) = \lim_{t \rightarrow 0^+} \frac{P_t^D \varphi_1(x) - \varphi_1(x)}{t} = \frac{e^{-\lambda_1(D)t} \varphi_1(x) - \varphi_1(x)}{t} = -\lambda_1(D) \varphi_1(x).$$

This gives that  $(A + \lambda_1(D))\varphi_1 = 0$  on  $D$ . It follows that  $\varphi_1$  is  $q$ -harmonic on  $B$  according to the  $\alpha$ -stable Schrödinger operator  $A + q$  with  $q \equiv \lambda_1(D)$ . Formally this follows from Proposition 3.17, Theorem 5.5, Definition 5.1 from [10] and the fact that  $(B, \lambda_1(D))$  is gaugeable because  $B$  is a proper open subset of  $D$  and  $\lambda_1(B) > \lambda_1(D)$ .

Let  $V_B(x, y) = \int_0^\infty e^{\lambda_1(D)t} p_B(t, x, y) dt$ . Here,  $V_B$  is the  $q$ -Green function, for  $q \equiv \lambda_1(D)$ , see page 58 in [10]. The  $q$ -harmonicity of  $\varphi_1$  (Definition 5.1 in [10]), Theorem 4.10 in [10] (formula (4.15)) and formula (2.17) in [10] (page 61) give that for  $z \in B$ ,

$$\begin{aligned} \varphi_1(z) &= E^z[e_{\lambda_1(D)}(\tau_B) \varphi_1(X(\tau_B))] \\ &= \mathcal{A}_{d,-\alpha} \int_B V_B(z, y) \int_{D \setminus B} |y - w|^{-d-\alpha} \varphi_1(w) dw dy, \end{aligned} \quad (3.6)$$

where  $e_{\lambda_1(D)}(\tau_B) = \exp(\lambda_1(D)\tau_B)$ . Of course (3.6) is a standard fact in the theory of  $q$ -harmonic functions for the  $\alpha$ -stable Schrödinger operators. For us this will be a key formula for proving the Harnack inequality for  $\varphi_1$ .

By the well known formula for the distribution of the harmonic measure [22] we have

$$E^z \varphi_1(X(\tau_B)) = \mathcal{A}_{d,-\alpha} \int_B G_B(z, y) \int_{D \setminus B} |y - w|^{-d-\alpha} \varphi_1(w) dw dy. \quad (3.7)$$

To obtain our Harnack inequality for  $\varphi_1$  we will first compare (3.6) and (3.7) and then we will use the formula for  $E^z \varphi_1(X(\tau_B))$ . In order to compare (3.6) and (3.7) we need to compare  $V_B(z, y)$  and  $G_B(z, y)$ . This will be done in a sequence of lemmas.

**Lemma 3.4.** *Let  $D \subset \mathbf{R}^d$ ,  $d > \alpha$  be a bounded domain with inradius 1 and  $B \subsetneq D$  be a ball with radius  $b < 1$ . Then for any  $z, y \in B$  and  $t_0 > 0$  we have*

$$V_B(z, y) \leq e^{\lambda_1(B_1)t_0} \int_0^{t_0} p_B(t, z, y) dt + \frac{C_3}{t_0^{(d-\alpha)/\alpha}},$$

where  $B_1 = B(0, 1)$  and  $C_3 = \alpha(d - \alpha)^{-1}(1 - b^\alpha)^{-d/\alpha} M_{d,\alpha}$ .

*Proof.* The inradius of  $D$  is 1 so  $\lambda_1(D) \leq \lambda_1(B_1)$ . It follows that

$$V_B(z, y) \leq e^{\lambda_1(B_1)t_0} \int_0^{t_0} p_B(t, z, y) dt + \int_{t_0}^\infty e^{\lambda_1(B_1)t} p_B(t, z, y) dt. \quad (3.8)$$

By (3.5) we obtain

$$p_B(t, z, y) = \sum_{n=1}^\infty e^{-\lambda_n(B)t} \varphi_{n,B}(z) \varphi_{n,B}(y) \leq \frac{1}{2} \sum_{n=1}^\infty e^{-\lambda_n(B)t} (\varphi_{n,B}^2(z) + \varphi_{n,B}^2(y)).$$

It follows that the second integral in (3.8) is bounded above by

$$\frac{1}{2} \int_{t_0}^\infty \sum_{n=1}^\infty e^{(\lambda_1(B_1) - \beta \lambda_n(B))t} e^{-\lambda_n(B)(1-\beta)t} (\varphi_{n,B}^2(z) + \varphi_{n,B}^2(y)) dt, \quad (3.9)$$

where  $\beta = \lambda_1(B_1)/\lambda_1(B) = b^\alpha$  (see 1.6).

Note also that  $e^{\lambda_1(B_1) - \beta \lambda_n(B)} \leq e^{\lambda_1(B_1) - \beta \lambda_1(B)} = e^0 = 1$ .

For any  $w \in B$  ( $w = z$  or  $w = y$ ) we have

$$\begin{aligned} \int_{t_0}^\infty \sum_{n=0}^\infty e^{-\lambda_n(B)(1-\beta)t} \varphi_{n,B}^2(w) dt &= \int_{t_0}^\infty p_B((1-\beta)t, w, w) dt \\ &\leq \int_{t_0}^\infty p((1-\beta)t, 0, 0) dt \leq \int_{t_0}^\infty \frac{M_{d,\alpha}}{(1-\beta)^{d/\alpha} t^{d/\alpha}} dt = \frac{C_3}{t_0^{(d-\alpha)/\alpha}}. \end{aligned}$$

□

**Lemma 3.5.** Let  $0 < a < b < 1$ ,  $B = B(w, b)$ ,  $w \in \mathbf{R}^d$ . For any  $y \in B$  and  $z \in B(w, a)$  we have

$$C_4 G_B(z, y) \geq E^y(\tau_B),$$

where  $C_4 = b^{d+\alpha/2} \alpha 2^{3d/2-\alpha/2-1} C_\alpha^d / ((b-a)^{\alpha/2} R_{d,\alpha} \mathcal{A}_{d,-\alpha})$ .

*Proof.* We may and do assume that  $w = 0$ . Let us consider the formula for the Green function for a unit ball  $G_{B(0,1)}(z, y)$  (3.3). Note that for any  $t > 0$

$$\int_0^t \frac{r^{\alpha/2-1} dr}{(r+1)^{d/2}} \geq \frac{1}{2^{d/2}} \int_0^{t \wedge 1} r^{\alpha/2-1} = \frac{(t^{\alpha/2} \wedge 1)}{\alpha 2^{d/2-1}}.$$

Hence for any  $z, y \in B(0, 1)$

$$G_{B(0,1)}(z, y) \geq R_{d,\alpha} \alpha^{-1} 2^{-d/2+1} |z-y|^{\alpha-d} (1 \wedge (w(z, y))^{\alpha/2}).$$

By scaling it follows that for any  $z, y \in B$ ,

$$\begin{aligned} G_B(z, y) &= b^{\alpha-d} G_{B(0,1)}\left(\frac{z}{b}, \frac{y}{b}\right) \\ &\geq \frac{R_{d,\alpha} \alpha^{-1} 2^{-d/2+1}}{b^{d-\alpha} \left|\frac{z}{b} - \frac{y}{b}\right|^{d-\alpha}} \left(1 \wedge \frac{\left(1 - \left|\frac{z}{b}\right|^2\right)^{\alpha/2} \left(1 - \left|\frac{y}{b}\right|^2\right)^{\alpha/2}}{\left|\frac{z}{b} - \frac{y}{b}\right|^\alpha}\right) \\ &= \frac{R_{d,\alpha} \alpha^{-1} 2^{-d/2+1}}{b^\alpha |z-y|^{d-\alpha}} \left(b^\alpha \wedge \frac{(b^2 - |z|^2)^{\alpha/2} (b^2 - |y|^2)^{\alpha/2}}{|z-y|^\alpha}\right). \end{aligned} \quad (3.10)$$

For  $z \in B(0, a)$  and  $y \in B = B(0, b)$  we have  $|z-y| \leq a+b \leq 2b$  and  $(b^2 - |z|^2)^{\alpha/2} \geq (b^2 - a^2)^{\alpha/2}$ . Hence

$$\frac{(b^2 - |z|^2)^{\alpha/2}}{|z-y|^\alpha} \geq \frac{((b-a)(b+a))^{\alpha/2}}{((a+b)^2)^{\alpha/2}} \geq \frac{1}{2^{\alpha/2}} \left(1 - \frac{a}{b}\right)^{\alpha/2}.$$

It follows that for  $z \in B(0, a)$  and  $y \in B(0, b)$ , (3.10) is bounded below by

$$\frac{R_{d,\alpha} \alpha^{-1} 2^{-d/2+1}}{b^d 2^{d-\alpha} 2^{\alpha/2}} \left(1 - \frac{a}{b}\right)^{\alpha/2} (b^2 - |y|^2)^{\alpha/2}.$$

By the formula for  $E^y(\tau_B)$  (3.2) this is equal to  $C_4^{-1} E^y(\tau_B)$ .  $\square$

**Lemma 3.6.** Let  $D \subset \mathbf{R}^d$ ,  $d > \alpha$  be a bounded domain with inradius 1,  $0 < a < b < 1$  and  $B = B(x, b) \subset D$ . Then for any  $z \in B(x, a)$  and  $y \in B$  we have  $G_B(z, y) \leq V_B(z, y) \leq C_5 G_B(z, y)$ , where  $C_5 = 1 + e + C_3 C_4 (\lambda_1(B_1))^{d/\alpha}$ .

*Proof.* The inequality  $G_B(z, y) \leq V_B(z, y)$  is trivial, it follows from the definition of  $G_B(z, y)$  and  $V_B(z, y)$ .

We will prove the inequality  $V_B(z, y) \leq C_5 G_B(z, y)$ . By Lemma 4.8 in [10] we have

$$V_B(z, y) = G_B(z, y) + \lambda_1(D) \int_B V_B(z, u) G_B(u, y) du. \quad (3.11)$$

By Lemma 3.4,  $\int_B V_B(z, u) G_B(u, y) du$  is bounded above by

$$e^{\lambda_1(B_1)t_0} \int_B \int_0^{t_0} p_B(t, z, u) dt G_B(u, y) du + \frac{C_3}{t_0^{(d-\alpha)/\alpha}} \int_B G_B(u, y) du. \quad (3.12)$$

Let us denote the above sum by I + II. We have

$$\begin{aligned} \int_B \int_0^{t_0} p_B(t, z, u) dt G_B(u, y) du &= \int_0^{t_0} \int_0^\infty \int_B p_B(t, z, u) p_B(s, u, y) du ds dt \\ &= \int_0^{t_0} \int_0^\infty p_B(t+s, z, y) ds dt \leq t_0 G_B(z, y). \end{aligned}$$

It follows that I  $\leq t_0 e^{\lambda_1(B_1)t_0} G_B(z, y)$ .

By applying Lemma 3.5 for  $z \in B(x, a)$  we get

$$\text{II} = \frac{C_3 E^y(\tau_B)}{t_0^{(d-\alpha)/\alpha}} \leq \frac{C_3 C_4 G_B(z, y)}{t_0^{(d-\alpha)/\alpha}}$$

Putting the estimates (3.11), (3.12) together with those for I and II gives

$$V_B(z, y) \leq G_B(z, y) \left( 1 + \lambda_1(B_1) t_0 e^{\lambda_1(B_1)t_0} + \frac{C_3 C_4 \lambda_1(B_1)}{t_0^{(d-\alpha)/\alpha}} \right). \quad (3.13)$$

Putting  $t_0 = 1/\lambda_1(B_1)$  we obtain

$$V_B(z, y) \leq G_B(z, y) (1 + e + C_3 C_4 (\lambda_1(B_1))^{d/\alpha}).$$

□

We now return to the proof of Theorem 3.3. Let  $z_1, z_2 \in B(x, a) \subset B(x, b) \subset D$ . By (3.6), (3.7) and Lemma 3.6 we obtain

$$\varphi_1(z_2) \geq E^{z_2}[\varphi_1(X(\tau_{B(x,b)}))] \quad (3.14)$$

and

$$\varphi_1(z_1) \leq C_5 E^{z_1}[\varphi_1(X(\tau_{B(x,b)}))]. \quad (3.15)$$

So to compare  $\varphi_1(z_2)$  and  $\varphi_1(z_1)$  we have to compare  $E^{z_1}[\varphi_1(X(\tau_{B(x,b)}))]$  and  $E^{z_2}[\varphi_1(X(\tau_{B(x,b)}))]$ .

We have

$$E^{z_i}[\varphi_1(X(\tau_{B(x,b)}))] = \int_{D \setminus \overline{B(x,b)}} \varphi_1(y) P_{b,x}(z_i, y) dy, \quad (3.16)$$

for  $i = 1, 2$ , where  $P_{b,x}(z_i, y)$  is the Poisson kernel for the ball  $B(x, b)$  which is given by an explicit formula (3.1). We have reduce to comparing  $P_{b,x}(z_1, y)$  and  $P_{b,x}(z_2, y)$ . Recall that  $z_1, z_2 \in B(x, a)$ . For  $y \in B^c(x, b)$  we have

$$\frac{|y - z_2|}{|y - z_1|} \leq \frac{b + a}{b - a}$$

and

$$\frac{(b^2 - |z_1 - x|^2)^{\alpha/2}}{(b^2 - |z_2 - x|^2)^{\alpha/2}} \leq \frac{b^\alpha}{(b^2 - a^2)^{\alpha/2}}.$$

It follows that

$$\frac{P_{b,x}(z_1, y)}{P_{b,x}(z_2, y)} \leq \frac{(b + a)^{d-\alpha/2} b^\alpha}{(b - a)^{d+\alpha/2}}.$$

Using this, (3.16), (3.15) and (3.14) we obtain for  $z_1, z_2 \in B(x, a)$

$$\varphi_1(z_1) \leq C_5 (b + a)^{d-\alpha/2} b^\alpha (b - a)^{-d-\alpha/2} \varphi_1(z_2).$$

□

In this paper we will need the Harnack inequality for  $\varphi_1^2$  in dimension  $d = 2$ . For this reason we will formulate the following corollary of Theorem 3.3. In this corollary we choose  $b \in (0, 1/2]$  and  $a = b/2$ .

**Corollary 3.7.** *Let  $\alpha \in (0, 2)$  and  $D \subset \mathbf{R}^2$  be a bounded domain with inradius  $R > 0$  and  $b \in (0, 1/2]$ . If  $B(x, bR) \subset D$  then on  $B(x, bR/2)$   $\varphi_1^2$  satisfies the Harnack inequality with constant  $c_H = c_H(\alpha)$ . That is, for any  $z_1, z_2 \in B(x, bR/2)$  we have  $\varphi_1^2(z_1) \leq c_H \varphi_1^2(z_2)$  where*

$$c_H = 3^{4-\alpha} 2^{2\alpha} \left( 4 + \frac{12\Gamma(2/\alpha)}{\alpha(2-\alpha)(1-2^{-\alpha})^{2/\alpha}} \right)^2. \quad (3.17)$$

We point out that  $c_H$  does not depend on  $b \in (0, 1/2]$ .

*Proof.* We are going to obtain upper bound estimates for constants  $C_1, C_2$  from Theorem 3.3 for  $d = 2$ ,  $a = b/2$  and  $b \in (0, 1/2]$ .

Putting  $d = 2$  we get  $C_\alpha^2 = \pi^{-2} \sin(\pi\alpha/2)$ ,  $R_{2,\alpha} = 2^{-\alpha} \pi^{-1} \Gamma^{-2}(\alpha/2)$ ,  $M_{2,\alpha} = 2^{-1} \pi^{-1} \alpha^{-1} \Gamma(2/\alpha)$ ,  $\mathcal{A}_{2,-\alpha} = \alpha^2 2^{\alpha-2} \pi^{-1} \Gamma(\alpha/2) \Gamma^{-1}(1 - \alpha/2)$ .

Putting these constants to the formula for  $C_2$  and using also the fact that  $\Gamma(\alpha/2)\Gamma(1-\alpha/2) = \pi \sin^{-1}(\pi\alpha/2)$  we obtain after easy calculations

$$C_2 = \frac{2^{3-\alpha/2}\Gamma(2/\alpha)(\lambda_1(B_1))^{2/\alpha}}{(2-\alpha)\alpha} \leq \frac{6 \cdot 2^{3-\alpha/2}\Gamma(2/\alpha)}{(2-\alpha)\alpha}.$$

The last inequality follows from (3.4) and the fact that  $\mu_1(B_1) < 6$ .

Putting  $d = 2$  and  $a = b/2$  we obtain

$$C_1 = 3^{2-\alpha/2}2^\alpha \left( 1 + e + \frac{2^{\alpha/2}b^2C_2}{(1-b^\alpha)^{2/\alpha}} \right).$$

Now using the estimate for  $C_2$  and the inequality  $b \leq 1/2$  we get

$$C_1 \leq 3^{2-\alpha/2}2^\alpha \left( 4 + \frac{12\Gamma(2/\alpha)}{\alpha(2-\alpha)(1-2^{-\alpha})^{2/\alpha}} \right). \quad (3.18)$$

In the assertion of Corollary 3.7 we have the Harnack inequality for  $\varphi_1^2$  so  $c_H$  is equal to the square of the right hand side of (3.18).  $\square$

## 4 Spectral gap for rectangles

We begin from several lemmas, which will lead us to the estimation of the spectral gap for rectangles.

**Lemma 4.1.** *Let  $D = (-L, L) \times (-1, 1)$ , where  $L \geq 1$ . Then*

$$\varphi_1(x) \leq \frac{3}{\sqrt{L}} \quad \text{for all } x \in D$$

and

$$\varphi_1(x_1, x_2) \geq \frac{1}{2\sqrt{L}}(1 - \frac{2}{L}|x_1|)(1 - 2|x_2|)$$

for all  $(x_1, x_2) \in [-L/2, L/2] \times [-1/2, 1/2]$ .

*Proof.* The lemma easily follows from unimodality and symmetry of  $\varphi_1$  (see Theorem 3.1), midconcavity of  $\varphi_1$  (see Theorem 1.1 in [5]) and the equality  $\int_D \varphi_1^2 dx = 1$ .  $\square$

**Lemma 4.2.** *Let  $\mu_k > 0$  ( $k = 1, \dots, L$ ),  $L \geq 2$  be unimodal, i.e., there exists  $k_0$  such that  $\mu_i \leq \mu_j$  for  $i \leq j \leq k_0$  and  $\mu_i \geq \mu_j$  for  $k_0 \leq i \leq j$ . Then for any  $f_k \in \mathbf{R}$  such that  $\sum_{k=1}^L f_k \mu_k = 0$  we have*

$$\sum_{k=1}^L \mu_k f_k^2 \leq L^2 \sum_{k=1}^{L-1} (\mu_k \wedge \mu_{k+1}) (f_k - f_{k+1})^2.$$

*Proof.* Let  $M = \sum_{k=1}^L \mu_k$ . By  $\sum_{k=1}^L \mu_k f_k = 0$  and Schwarz inequality we obtain

$$\begin{aligned}
M \sum_{k=1}^L \mu_k f_k^2 &= \sum_{j=1}^L \mu_j \sum_{k=1}^L \mu_k f_k^2 = \frac{1}{2} \sum_{j,k=1}^L \mu_j \mu_k (f_j^2 + f_k^2) \\
&= \frac{1}{2} \sum_{j,k=1}^L \mu_j \mu_k (f_j - f_k)^2 \\
&= \sum_{1 \leq j < k \leq L} \mu_j \mu_k \left( \sum_{t=j}^{k-1} (f_t - f_{t+1}) \right)^2 \\
&\leq L \sum_{1 \leq j < k \leq L} \mu_j \mu_k \sum_{t=j}^{k-1} (f_t - f_{t+1})^2 \\
&= L \sum_{t=1}^{L-1} \left( \sum_{j \leq t < k} \mu_j \mu_k \right) \cdot (f_t - f_{t+1})^2. \tag{4.1}
\end{aligned}$$

For  $t < k_0$  (where  $k_0$  is defined in the lemma) we have

$$\sum_{j \leq t < k} \mu_j \mu_k \leq L \mu_t \sum_{k=1}^L \mu_k = L(\mu_t \wedge \mu_{t+1})M.$$

Similarly for  $t \geq k_0$

$$\sum_{j \leq t < k} \mu_j \mu_k \leq L \mu_{t+1} \sum_{j=1}^L \mu_j = L(\mu_t \wedge \mu_{t+1})M.$$

These two inequalities combined with (4.1) finish the proof.  $\square$

**Lemma 4.3.** *Let  $(D, \mu)$  be a finite measure space and  $D = \bigcup_{k=1}^L D_k$ ,  $L \geq 1$  with pairwise disjoint  $D_k$ 's. We assume that the sequence  $\mu_k = \mu(D_k) > 0$  is unimodal. Then*

$$\frac{1}{\mu(D)} \int_D \int_D (f(x) - f(y))^2 \mu(dx) \mu(dy) \tag{4.2}$$

$$\leq 2 \sum_{k=1}^L \frac{1}{\mu_k} \int_{D_k} \int_{D_k} (f(x) - f(y))^2 \mu(dx) \mu(dy) \tag{4.3}$$

$$+ 4L^2 \sum_{k=1}^{L-1} \frac{1}{\mu_k \vee \mu_{k+1}} \int_{D_k} \int_{D_{k+1}} (f(x) - f(y))^2 \mu(dx) \mu(dy) \tag{4.4}$$

for all  $f \in L^2(D, \mu)$ .

*Proof.* Let  $f \in L^2(D, \mu)$ . Without loss of generality we may assume that  $L \geq 2$  and  $\int_D f d\mu = 0$ . Then (4.2) is equal to  $2 \int_D f^2 d\mu$ .

Let  $f_k = \frac{1}{\mu_k} \int_{D_k} f d\mu$ . We have

$$\sum_{k=1}^L \frac{1}{\mu_k} \int_{D_k} \int_{D_k} (f(x) - f(y))^2 \mu(dx) \mu(dy) = 2 \int_D f^2 d\mu - 2 \sum_{k=1}^L \mu_k f_k^2.$$

Thus if  $\sum_{k=1}^L \mu_k f_k^2 \leq \frac{1}{2} \int_D f^2 d\mu$ , then (4.2 - 4.4) holds. Consequently, from now on we may assume that

$$\sum_{k=1}^L \mu_k f_k^2 > \frac{1}{2} \int_D f^2 d\mu. \quad (4.5)$$

Thus, by Lemma 4.2 we have

$$2 \int_D f^2 d\mu < 4L^2 \sum_{k=1}^{L-1} (\mu_k \wedge \mu_{k+1}) (f_k - f_{k+1})^2. \quad (4.6)$$

On the other hand,

$$\begin{aligned} & \int_{D_k} \int_{D_{k+1}} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\ &= \int_{D_k} \int_{D_{k+1}} ((f(x) - f_k) - (f(y) - f_{k+1}) + (f_k - f_{k+1}))^2 \mu(dx) \mu(dy) \\ &\geq \mu_k \mu_{k+1} (f_k - f_{k+1})^2. \end{aligned}$$

The lemma now follows from (4.6).  $\square$

**Lemma 4.4.** Let  $\alpha \in [1, 2)$ ,  $D = (-L, L) \times (-1, 1)$ ,  $L \geq 1$  and  $\varphi_1$  be the first eigenfunction for  $\{P_t^D\}_{t \geq 0}$ . Let  $-L + 1/4 \leq a \leq b \leq L - 1/4$ ,  $b - a = 1/8$  and put  $A = [a, b] \times [-1/8, 1/8]$ . Then we have

$$\left( \max_{x \in A} \varphi_1^2(x) \right) \left( \int_A \varphi_1^2(x) dx \right)^{-1} \leq C_R, \quad (4.7)$$

where  $C_R = 10^4$ .

*Proof.* We will use the fact that  $\varphi_1$  is symmetric and unimodal with respect to both coordinate axes (see Theorem 3.1). We will also use much stronger fact that  $\varphi_1$  is "midconcave" (see Theorem 1.1 in [5]). That is for any  $x_2 \in (-1, 1)$   $x_1 \rightarrow \varphi_1(x_1, x_2)$  is concave on  $(-L/2, L/2)$  and for any  $x_1 \in (-L, L)$   $x_2 \rightarrow \varphi_1(x_1, x_2)$  is concave on  $(-1/2, 1/2)$ .

By symmetry of  $\varphi_1$  we may and do assume that  $b \geq 0$ . We will consider two cases: Case 1,  $b \in [0, 3/8]$ , Case 2,  $b \in [3/8, L]$ .

At first let us consider Case 1:  $b \in [0, 3/8]$ . Note that by the unimodality  $\min_{x \in A} \varphi_1(x)$  is equal to  $\varphi_1(b, 1/8)$  or  $\varphi_1(a, 1/8)$ . By concavity of  $x_2 \rightarrow \varphi_1(b, x_2)$  on  $(-1/2, 1/2)$  we obtain

$$\varphi_1(b, 1/8) \geq (3/4)\varphi_1(b, 0) + (1/4)\varphi_1(b, 1/2) \geq (3/4)\varphi_1(b, 0).$$

On the other hand  $x_1 \rightarrow \varphi_1(x_1, 0)$  is concave on  $(-L/2, L/2)$ . We have  $L \geq 1$  so  $x_1 \rightarrow \varphi_1(x_1, 0)$  is concave on  $(-1/2, 1/2)$ . It follows that

$$\varphi_1(b, 0) \geq \varphi_1(3/8, 0) \geq (1/4)\varphi_1(0, 0) + (3/4)\varphi_1(1/2, 0) \geq (1/4)\varphi_1(0, 0).$$

Hence  $\varphi_1(b, 1/8) \geq (3/16)\varphi_1(0, 0)$ . Similarly one can show that  $\varphi_1(a, 1/8) \geq (3/16)\varphi_1(0, 0)$ .

We also have  $\max_{x \in D} \varphi_1(x) = \varphi_1(0, 0) \geq \max_{x \in A} \varphi_1(x)$ . Finally

$$\int_A \varphi_1^2(x) dx \geq |A| \min_{x \in A} \varphi_1^2(x) \geq \frac{1}{32} \left( \frac{3}{16} \right)^2 \varphi_1^2(0, 0) \geq \frac{9}{2^{13}} \max_{x \in A} \varphi_1^2(x).$$

This gives (4.7) and finishes Case 1.

Now let us consider Case 2:  $b \in [3/8, L]$ . Note that  $\max_{x \in A} \varphi_1(x) = \varphi_1(a, 0)$  and  $\min_{x \in A} \varphi_1(x) = \varphi_1(b, 1/8)$ . As before  $\varphi_1(b, 1/8) \geq (3/4)\varphi_1(b, 0)$ . Hence

$$\int_A \varphi_1^2(x) dx \geq |A| \left( \frac{3}{4} \right)^2 \varphi_1^2(b, 0) = \frac{9}{2^9} \varphi_1^2(b, 0). \quad (4.8)$$

Now we have to estimate  $\varphi_1(b, 0)$ .

Let  $x_0 = (b, 0)$ ,  $r = \sqrt{2}/8$  and consider a ball  $B = B(x_0, r)$ . It is easy to note that  $\overline{B} \subset D$ . By formula (3.6) and the fact that  $e_{\lambda_1(D)}(\tau_B) \geq 1$  we have

$$\varphi_1(x_0) \geq E^{x_0}[\varphi_1(X(\tau_B))]. \quad (4.9)$$

Now let us introduce polar coordinates  $(\rho, \psi)$  with centre at  $x_0 = (b, 0)$ . For any  $z = (z_1, z_2) \in \mathbf{R}^2$  we have  $z_1 - b = \rho \cos(\psi)$ ,  $z_2 = \rho \sin(\psi)$ . Let us consider the set  $S_1 = \{(\rho, \psi) : \rho \in (r, 2r), \psi \in (3\pi/4, 5\pi/4)\}$ . Note that  $S_1$  is chosen so that  $S_1 \subset [b-2r, b-r/\sqrt{2}] \times [-\sqrt{2}r, \sqrt{2}r] \subset [b-3/8, a] \times [-1/4, 1/4]$ .

By unimodality and "midconcavity" for any  $z \in [b-3/8, a] \times [-1/4, 1/4]$  we have  $\varphi_1(z) \geq \varphi_1(a, 0)/2$ . This and (4.9) gives

$$\varphi_1(x_0) \geq E^{x_0}[\varphi_1(X(\tau_B)); X(\tau_B) \in S_1] \geq (\varphi_1(a, 0)/2) P^{x_0}(X(\tau_B) \in S_1). \quad (4.10)$$

We have

$$P^{x_0}(X(\tau_B) \in S_1) = \int_{S_1} P_{\tau, x_0}(x_0, y) dy,$$

where  $P_{r,x_0}(x_0, y)$  is the Poisson kernel for  $B$  given by (3.1).

Let  $S_2 = \{(\rho, \psi) : \rho \in (2r, \infty), \psi \in (3\pi/4, 5\pi/4)\}$ . Since  $P^{x_0}(X(\tau_B) \in B^c) = 1$  and the distribution  $P^{x_0}(X(\tau_B) \in \cdot)$  is invariant under rotation around  $x_0$  it is easy to note that  $P^{x_0}(X(\tau_B) \in S_1 \cup S_2) = 1/4$ . Hence

$$P^{x_0}(X(\tau_B) \in S_1) = \frac{1}{4} - P^{x_0}(X(\tau_B) \in S_2) = \frac{1}{4} - \int_{S_2} P_{r,x_0}(x_0, y) dy. \quad (4.11)$$

We have

$$\int_{S_2} P_{r,x_0}(x_0, y) dy = C_\alpha^2 \int_{3\pi/4}^{5\pi/4} \int_{2r}^{\infty} \frac{r^\alpha}{(\rho^2 - r^2)^{\alpha/2} \rho^2} \rho d\rho d\psi. \quad (4.12)$$

Note that  $\rho^2 - r^2 \geq (3/4)\rho^2$  for  $\rho \geq 2r$  so (4.12) is smaller than

$$C_\alpha^2 \frac{\pi}{2} r^\alpha \left(\frac{4}{3}\right)^{\alpha/2} \int_{2r}^{\infty} \rho^{-\alpha-1} d\rho = \frac{1}{2\pi\alpha 3^{\alpha/2}} \sin\left(\frac{\pi\alpha}{2}\right) \leq \frac{1}{2\pi\sqrt{3}}.$$

The last inequality follows from the fact that in this lemma we assume that  $\alpha \in [1, 2)$ .

Using this, (4.10) and (4.11) we obtain

$$\varphi_1(x_0) \geq \frac{\varphi_1(a, 0)}{2} \left( \frac{1}{4} - \frac{1}{2\pi\sqrt{3}} \right).$$

This and (4.8) gives

$$\int_A \varphi_1^2(x) dx \geq \frac{9}{29} \left( \frac{1}{2} \left( \frac{1}{4} - \frac{1}{2\pi\sqrt{3}} \right) \right)^2 \varphi_1^2(a, 0) \geq \frac{\varphi_1^2(a, 0)}{10^4} = C_R^{-1} \max_{x \in A} \varphi_1^2(x).$$

□

*Proof of Theorem 1.2 – part I.* By scaling of eigenvalues (see (1.6)) it is sufficient to show the following inequalities for rectangles  $D = (-L, L) \times (-1, 1)$ ,  $L \geq 1$ :

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \leq 10^6 \cdot \begin{cases} \frac{2}{1-\alpha} \frac{1}{L^{1+\alpha}} & \text{for } \alpha < 1, \\ \frac{2 \log(L+1)}{L^2} & \text{for } \alpha = 1, \\ \left( \frac{1}{2-\alpha} + \frac{1}{\alpha-1} \right) \frac{1}{L^2} & \text{for } \alpha > 1. \end{cases} \quad (4.13)$$

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \geq \begin{cases} \frac{1}{36 \cdot 2^{1+2\alpha}(L)^{1+\alpha}} & \text{for } \alpha < 1, \\ 10^{-9} \frac{\log(L+1)}{L^2} & \text{for } \alpha = 1, \\ \frac{1}{33 \cdot 13^{1+\alpha/2} \cdot 10^4} \frac{1}{L^2} & \text{for } \alpha > 1. \end{cases} \quad (4.14)$$

Similarly, to prove Remark 1.3 it is sufficient to show

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \geq \frac{1}{36 \cdot 2^\alpha (L+1)^{1+\alpha}}. \quad (4.15)$$

Let us take  $f(x_1, x_2) = x_1$  for  $x = (x_1, x_2) \in D$ . Then by Lemma 4.1

$$\begin{aligned} \int_D f^2 \varphi_1^2 dx &\geq \frac{1}{4L} \int_{-L/2}^{L/2} \int_{-1/2}^{1/2} (1 - \frac{2}{L}|x_1|)^2 (1 - 2|x_2|)^2 x_1^2 dx_2 dx_1 \\ &= \frac{L^2}{1440}. \end{aligned}$$

On the other hand, for  $x \in D$  we have

$$\begin{aligned} \int_D |x - y|^{-\alpha} dy &\leq \int_{B(0, \sqrt{5})} |y|^{-\alpha} dy \\ &+ \int_{-L}^L \int_{-1}^1 (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{-\alpha/2} \mathbf{1}_{\mathbf{R} \setminus [-1, 1]}(y_1) dy_2 dy_1 \\ &\leq 2\pi \frac{5^{1-\alpha/2}}{2-\alpha} + 4 \int_1^L y_1^{-\alpha} dy_1. \end{aligned}$$

Thus by Lemma 4.1

$$\mathcal{E}(f, f) < \frac{A_{2,-\alpha}}{2} \frac{9}{L} \cdot 4L \cdot (2\pi \frac{5^{1-\alpha/2}}{2-\alpha} + 4 \int_1^L y_1^{-\alpha} dy_1).$$

Hence by Theorem 1.1

$$\begin{aligned} \lambda_2 - \lambda_1 &\leq \frac{\mathcal{E}(f, f)}{\int_D f^2 \varphi_1^2 dx} \\ &< \frac{A_{2,-\alpha}}{2} \frac{72 \cdot 1440}{L^2} (\pi \frac{5^{1-\alpha/2}}{2-\alpha} + 2 \int_1^L y_1^{-\alpha} dy_1), \end{aligned}$$

therefore (4.13) is proven.

For  $f \in L^2(D, \varphi_1^2)$  such that  $\int_D f \varphi_1^2 dx = 0$  we have by Lemma 4.1

$$\begin{aligned} \int_D f^2 \varphi_1^2 dx &= \frac{1}{2} \int_D \int_D (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy \\ &\leq \frac{9 \operatorname{diam}(D)^{2+\alpha}}{2L} \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{2+\alpha}} \varphi_1(x) \varphi_1(y) dx dy \\ &\leq 36 \cdot 2^\alpha (L + 1)^{1+\alpha} \frac{2}{A_{2,-\alpha}} \mathcal{E}(f, f), \end{aligned}$$

thus by Theorem 1.1 we obtain (4.15) and also (4.14) in the case when  $\alpha < 1$ .

Let  $D = \bigcup_{k=1}^{[2L]} D_k$  be divided into  $[2L]$  pairwise disjoint rectangles  $D_k$  of size  $\frac{2L}{[2L]} \times 2$ , denote  $E_k = D_k \cup D_{k+1}$ . Let  $\mu = \varphi_1^2 dx$ , by Theorem 3.1 we see that  $(D, \mu)$  satisfies the assumptions of Lemma 4.3. Thus for  $f \in L^2(D, \varphi_1^2)$  such that  $\int_D f \varphi_1^2 dx = 0$  we have

$$\begin{aligned} 2 \int_D f^2 \varphi_1^2 dx &\leq 2 \sum_{k=1}^{[2L]} \frac{1}{\int_{D_k} \varphi_1^2 dx} \int_{D_k} \int_{D_k} (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy \\ &+ 4 [2L]^2 \sum_{k=1}^{[2L]-1} \frac{2}{\int_{E_k} \varphi_1^2 dx} \int_{E_k} \int_{E_k} (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy \\ &= I_1 + I_2, \end{aligned}$$

$$\begin{aligned} I_2 &\leq 32L^2 \sum_{k=1}^{[2L]-1} \frac{\sup_{E_k} \varphi_1^2}{\int_{E_k} \varphi_1^2 dx} \\ &\quad \times \int_{E_k} \int_{E_k} (f(x) - f(y))^2 \frac{\operatorname{diam}(E_k)^{2+\alpha}}{|x - y|^{2+\alpha}} \varphi_1(x) \varphi_1(y) dx dy. \end{aligned}$$

Hence by Lemma 4.4 and  $\operatorname{diam}(E_k) \leq \sqrt{13}$  we obtain

$$I_2 \leq 64 \cdot 13^{1+\alpha/2} C_R L^2 \frac{2}{A_{2,-\alpha}} \mathcal{E}(f, f).$$

Similarly,

$$I_1 \leq 2 \cdot 13^{1+\alpha/2} C_R \frac{2}{A_{2,-\alpha}} \mathcal{E}(f, f)$$

and (4.14) in the case when  $\alpha > 1$  follows.  $\square$

*Proof of Theorem 1.2 – part II, the case  $\alpha = 1$ .* Let  $N = [L]$ . We divide  $D$  into  $2N$  rectangles of equal size  $D_{-N+1}, \dots, D_N$ , where

$$D_k = ((k-1)L/N, kL/N) \times (-1, 1), \quad k = -N+1, \dots, N.$$

Let us note that Lemma 4.4 implies

$$\left( \sup_{x \in D_k} \varphi_1^2 \right) \left( \int_{D_k} \varphi_1^2 \right)^{-1} \leq C_1^{-1},$$

where  $C_1 = C_R^{-1} = 10^{-4}$ .

We will also use the following easy inequality  $\inf_{x \in D_k, y \in D_{k+1}} |x-y|^{-3} \geq C_2$ , where  $C_2 = (\sqrt{20})^{-3}$ .

In the case  $\alpha = 1$  we will show that

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq \frac{C_1 C_2 \log(L+1)}{360 L^2}. \quad (4.16)$$

This implies (4.14) in the remaining case when  $\alpha = 1$ .

Fix  $i \in \{1, \dots, N\}$ . For any  $k = 1, \dots, i$  let

$$A_k^i = \dots \cup D_{k-2i} \cup D_{k-i} \cup D_k \cup D_{k+i} \cup D_{k+2i} \cup \dots$$

Since  $N$  is not necessarily divisible by  $i$  the number of “parts” of  $A_k^i$  may not be equal for different  $k$ . To make the definition of  $A_k^i$  more precise we introduce some more notation.

We have  $N = i[N/i] + r(i)$  for some  $r(i) \in \{0, \dots, i-1\}$ . Let  $q(i, k) = [N/i]$  for  $k = 1, \dots, r(i)$ ,  $q(i, k) = [N/i] - 1$  for  $k = r(i) + 1, \dots, i$  and  $p(i, k) = -[N/i]$  for  $k = 1, \dots, i - r(i)$ ,  $p(i, k) = -[N/i] - 1$  for  $k = i - r(i) + 1, \dots, i$ .

For  $m = p(i, k), \dots, q(i, k)$  let  $D_{k,m}^i = D_{k+mi}$ . Then we have

$$A_k^i = \bigcup_{m=p(i,k)}^{q(i,k)} D_{k,m}^i.$$

Now we will apply Lemma 4.1 to the set  $A_k^i$  which is divided as above. We take  $\mu(dx) = \varphi_1^2(x) dx$  and  $f = \varphi_2/\varphi_1$ . Let us denote  $\mu_{k,m}^i = \int_{D_{k,m}^i} \varphi_1^2$ .

Of course we have  $\mu_{k,m}^i \vee \mu_{k,m+1}^i \geq (\mu_{k,m}^i \mu_{k,m+1}^i)^{1/2}$  and

$$\int_{A_k^i} \int_{A_k^i} (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy = 2 \int_{A_k^i} f^2 \varphi_1^2 \int_{A_k^i} \varphi_1^2 - 2 \left( \int_{A_k^i} f \varphi_1^2 \right)^2.$$

So applying Lemma 4.1 to  $A_k^i$  and summing from  $k = 1$  to  $k = i$  we

obtain

$$2 \sum_{k=1}^i \int_{A_k^i} f^2 \varphi_1^2 - 2 \sum_{k=1}^i \left( \int_{A_k^i} \varphi_1^2 \right)^{-1} \left( \int_{A_k^i} f \varphi_1^2 \right)^2 \leq \quad (4.17)$$

$$2 \sum_{k=1}^i \sum_{m=p(i,k)}^{q(i,k)} \frac{1}{\mu_{k,m}^i} \int_{D_{k,m}^i} \int_{D_{k,m}^i} (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy \quad (4.18)$$

$$+ 4 \sum_{k=1}^i \sum_{m=p(i,k)}^{q(i,k)-1} \frac{(q(i,k) - p(i,k) + 1)^2}{(\mu_{k,m}^i \mu_{k,m+1}^i)^{1/2}} \int_{D_{k,m}^i} \int_{D_{k,m+1}^i} \times (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy. \quad (4.19)$$

Now we will consider 2 cases:

Case 1. For any  $i \in \{1, \dots, [N^{1/4}]\}$  we have

$$\sum_{k=1}^i \left( \int_{A_k^i} \varphi_1^2 \right)^{-1} \left( \int_{A_k^i} f \varphi_1^2 \right)^2 \leq \frac{1}{2}. \quad (4.20)$$

Case 2. There exists  $i \in \{1, \dots, [N^{1/4}]\}$  such that

$$\sum_{k=1}^i \left( \int_{A_k^i} \varphi_1^2 \right)^{-1} \left( \int_{A_k^i} f \varphi_1^2 \right)^2 > \frac{1}{2}. \quad (4.21)$$

At first we consider Case 1. Let us denote expressions in (4.17), (4.18), (4.19) by  $L(i)$ ,  $R(i)$ ,  $S(i)$  respectively.

We have  $\sum_{k=1}^i \int_{A_k^i} f^2 \varphi_1^2 = \int_D f^2 \varphi_1^2 = 1$  so by the assumption (4.20) we have  $L(i) \geq 1$ .

Now let us assume that for some  $i \in \{1, \dots, [N^{1/4}]\}$  we have  $R(i) \geq S(i)$ . This gives  $R(i) \geq L(i)/2 \geq 1/2$ . On the other hand we have

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) = \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy \quad (4.22)$$

$$\geq \sum_{k=1}^i \sum_{m=p(i,k)}^{q(i,k)} \int_{D_{k,m}^i} \int_{D_{k,m}^i} \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy. \quad (4.23)$$

By our standard arguments (4.23) is bounded below by

$$C_1 C_2 \sum_{k=1}^i \sum_{m=p(i,k)}^{q(i,k)} (\mu_{k,m}^i)^{-1} \int_{D_{k,m}^i} \int_{D_{k,m}^i} (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy.$$

This is equal to  $(C_1 C_2 / 2) R(i)$  where  $R(i)$  is the expression in (4.18). Since  $R(i) \geq 1/2$ , (4.22 - 4.23) gives

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq \frac{C_1 C_2 R(i)}{2} \geq \frac{C_1 C_2}{4} \geq \frac{C_1 C_2 \log(L+1)}{4L^2},$$

which proves (4.16).

So now we assume that for all  $i \in \{1, \dots, [N^{1/4}]\}$  we have  $R(i) < S(i)$ . This gives  $S(i) \geq L(i)/2 \geq 1/2$ .

Let us observe that

$$\begin{aligned} 2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) &= \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy \geq \\ &\sum_{i=1}^{[N^{1/4}]} \sum_{k=1}^i \sum_{m=p(i,k)}^{q(i,k)-1} \int_{D_{k,m}^i} \int_{D_{k,m+1}^i} \frac{(f(x) - f(y))^2}{|x - y|^3} \\ &\quad \times \varphi_1(x) \varphi_1(y) dx dy. \end{aligned} \quad (4.24)$$

Note that

$$\sup_{x \in D_{k,m}^i, y \in D_{k,m+1}^i} |x - y|^3 \leq ((2i+2)^2 + 2^2)^{3/2} \leq (20i^2)^{3/2} = C_2 i^3.$$

So by our standard arguments (4.24) is bounded below by

$$\sum_{i=1}^{[N^{1/4}]} \frac{C_1 C_2}{i^3} \sum_{k=1}^i \sum_{m=p(i,k)}^{q(i,k)-1} (\mu_{k,m}^i \mu_{k,m+1}^i)^{-1/2} \int_{D_{k,m}^i} \int_{D_{k,m+1}^i} \quad (4.25)$$

$$\times (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy. \quad (4.26)$$

Note that  $|q(i,k) - p(i,k) + 1| \leq 2N/i + 1 \leq 3N/i$ . Hence (4.25 - 4.26) is bounded below by

$$\sum_{i=1}^{[N^{1/4}]} \frac{C_1 C_2}{i^3} \left( \frac{i}{3N} \right)^2 \frac{S(i)}{4},$$

where  $S(i)$  is the expression in (4.19). We assumed that  $S(i) \geq 1/2$ . Therefore

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq \frac{C_1 C_2}{2^3 \cdot 3^2 N^2} \sum_{i=1}^{[N^{1/4}]} \frac{1}{i} \geq \frac{C_1 C_2 \log([N^{1/4}] + 1)}{2^3 \cdot 3^2 N^2}.$$

Note that  $([N^{1/4}] + 1)^5 \geq 2([N^{1/4}] + 1)^4 \geq 2N \geq N + 1$ . Hence  $\log([N^{1/4}] + 1) \geq (\log(N + 1))/5$ . Note also that  $L \geq N$  and a function  $\log(x + 1)/x^2$  is decreasing for  $x \geq 1$ . Therefore

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq \frac{C_1 C_2 \log(N + 1)}{2^3 \cdot 3^2 \cdot 5 N^2} \geq \frac{C_1 C_2 \log(L + 1)}{360 L^2}.$$

This shows (4.16) and finishes Case 1.

Now let us consider Case 2. In this case we will show the following lemma.

**Lemma 4.5.** *If  $N \geq 16$  and there exist  $i \in \{1, \dots, [N^{1/4}]\}$  such that*

$$\sum_{k=1}^i \left( \int_{A_k^i} \varphi_1^2 \right)^{-1} \left( \int_{A_k^i} f \varphi_1^2 \right)^2 > \frac{1}{2} \quad (4.27)$$

then

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq 2C_1C_2 \left( \frac{1}{256i^3} - \frac{72}{N} \right).$$

Before we come to the proof of this lemma (which is quite technical) let us first show how this lemma implies (4.16).

We know (Case 2) that (4.27) holds for some  $i \in \{1, \dots, [N^{1/4}]\}$ . Hence for  $N \geq 16$  we have

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq \frac{2C_1C_2}{256N} \left( \frac{N}{i^3} - 72 \cdot 256 \right) \geq \frac{2C_1C_2}{256N} (N^{1/4} - 18432). \quad (4.28)$$

When (say)  $N \geq 10^{18}$  then  $N^{1/4} \geq 3 \cdot 10^4$  and (4.28) implies (4.16).

When  $N \leq 10^{18}$  we have  $\log(L+1) \leq \log(N+2) \leq 42$ . Then Remark 1.3 implies

$$2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) \geq \frac{1}{72(L+1)^2} \geq \frac{\log(L+1)}{72 \cdot 4L^2 \cdot 42},$$

which also gives (4.16).  $\square$

*Proof of Lemma 4.5.* Note that if  $i = 1$  then the left hand side of (4.27) equals 0. So we may and do assume that  $i \geq 2$ .

In this proof  $i \in \{2, \dots, [N^{1/4}]\}$  is fixed so we will drop  $i$  from the notation. We will write  $D_{k,m}$  for  $D_{k,m}^i$ ,  $A_k$  for  $A_k^i$ ,  $p(k)$ ,  $q(k)$  for  $p(i, k)$ ,  $q(i, k)$ . We will also introduce the following notation

$$a_{k,m} = \int_{D_{k,m}} f \varphi_1^2 \left( \int_{D_{k,m}} \varphi_1^2 \right)^{-1}, \quad b_{k,m} = \int_{D_{k,m}} \varphi_1^2,$$

$$a_k = \int_{A_k} f \varphi_1^2 = \sum_{m=p(k)}^{q(k)} a_{k,m} b_{k,m}, \quad b_k = \int_{A_k} \varphi_1^2 = \sum_{m=p(k)}^{q(k)} b_{k,m}.$$

The condition (4.27) written in our notation is

$$\sum_{k=1}^i \frac{a_k^2}{b_k} > \frac{1}{2}. \quad (4.29)$$

Now we have to estimate  $b_k$  from below. Note that  $b_1 + \dots + b_i = 1$ . Roughly speaking, since  $\varphi_1$  is "midconcave", for  $N$  large enough  $b_1, \dots, b_i$  have similar values so  $b_k \geq c/i$  for  $k = 1, \dots, i$ . The following lemma makes the above remark precise.

**Lemma 4.6.** *For  $N \geq 16$  and any  $k = 1, \dots, i$  we have*

$$b_k \geq 1/(32i). \quad (4.30)$$

*Proof.* Note that  $\sum_{l=1}^N \int_{D_l} \varphi_1^2 = 1/2$  and  $\int_{D_l} \varphi_1^2$  is nonincreasing in  $l$  ( $l = 1, \dots, N$ ) so  $\int_{D_1} \varphi_1^2 \geq 1/(2N)$ .

For any  $x_2 \in (-1, 1)$  the function  $x_1 \rightarrow \varphi_1(x_1, x_2)$  is concave for  $x_1 \in [-L/2, L/2]$  and attains its maximum for  $x_1 = 0$ . Hence, for any  $x_2 \in (-1, 1)$  and  $x_1 \in [-L/4, L/4]$  we have  $\varphi_1(x_1, x_2) \geq \varphi_1(0, x_2)/2$ .

Recall that  $D_l = ((l-1)L/N, lL/N) \times (-1, 1)$ . If  $l \in [-N/4 + 1, N/4]$  then  $(l-1)L/N \geq -L/4$  and  $lL/N \leq L/4$ . It follows that for such  $l$

$$\int_{D_l} \varphi_1^2(x_1, x_2) dx_1 dx_2 \geq \frac{1}{4} \int_{D_l} \varphi_1^2(0, x_2) dx_1 dx_2 \geq \frac{1}{4} \int_{D_1} \varphi_1^2 \geq \frac{1}{8N}.$$

Recall that  $A_k = \bigcup_{m=p(k)}^{q(k)} D_{k,m} = \bigcup_{m=p(k)}^{q(k)} D_{k+mi}$ .

Let  $C_k = \{m \in \mathbf{Z} : -N/4 + 1 \leq k + mi \leq N/4\}$ . For any  $m \in C_k$  we have  $\int_{D_{k+mi}} \varphi_1^2 \geq 1/(8N)$  so

$$b_k = \int_{A_k} \varphi_1^2 = \sum_{m=p(k)}^{q(k)} \int_{D_{k+mi}} \varphi_1^2 \geq \frac{\#C_k}{8N}, \quad (4.31)$$

where  $\#C_k$  is the number of elements of  $C_k$ . We have

$$\#C_k \geq \left\lceil \frac{2[N/4]}{i} \right\rceil \geq \frac{2((N/4) - 1)}{i} - 1 = \frac{N - 4 - 2i}{2i}.$$

We have  $i \leq N^{1/4}$  and  $N \geq 16$  so it is not difficult to show that  $N - 4 - 2i \geq N/2$ . Hence  $\#C_k \geq N/(4i)$ . Finally this and (4.31) gives (4.30).  $\square$

By (4.29) and Lemma 4.6 we obtain  $\sum_{k=1}^i a_k^2 \geq 1/(64i)$ .

Note that  $\sum_{k=1}^i a_k = \int_D f \varphi_1^2 = 0$ . Therefore using Lemma 4.2 we get

$$\sum_{k=1}^{i-1} (a_{k+1} - a_k)^2 \geq \frac{1}{i^2} \sum_{k=1}^i a_k^2 \geq \frac{1}{64i^3}. \quad (4.32)$$

We have

$$\begin{aligned} 2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) &= \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy \\ &\geq 2 \sum_{k=1}^{i-1} \int_{A_k} \int_{A_{k+1}} \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy. \end{aligned}$$

By our definition of  $p(k)$  and  $q(k)$  it is easy to notice that  $p(k+1) \leq p(k)$  and  $q(k+1) \leq q(k)$ .

It follows that

$$\begin{aligned} &\int_{A_k} \int_{A_{k+1}} \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy \\ &\geq \sum_{m=p(k)}^{q(k+1)} \int_{D_{k,m}} \int_{D_{k+1,m}} \frac{(f(x) - f(y))^2}{|x - y|^3} \varphi_1(x) \varphi_1(y) dx dy. \end{aligned}$$

By our standard arguments this is bounded below by

$$\begin{aligned} &\sum_{m=p(k)}^{q(k+1)} \frac{C_1 C_2}{b_{k,m}^{1/2} b_{k+1,m}^{1/2}} \int_{D_{k,m}} \int_{D_{k+1,m}} (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy \\ &\geq C_1 C_2 \sum_{m=p(k)}^{q(k+1)} (a_{k,m} - a_{k+1,m})^2 b_{k,m}^{1/2} b_{k+1,m}^{1/2}. \end{aligned}$$

The last inequality follows from the argument which has been already used in the last 3 lines in the proof of Lemma 4.3. By Schwarz inequality it is bounded below by

$$C_1 C_2 \left( \sum_{m=p(k)}^{q(k+1)} (a_{k,m} - a_{k+1,m}) b_{k,m}^{1/2} b_{k+1,m}^{1/2} \right)^2 \left( \sum_{m=p(k)}^{q(k+1)} b_{k,m}^{1/2} b_{k+1,m}^{1/2} \right)^{-1}. \quad (4.33)$$

We have

$$\sum_{m=p(k)}^{q(k+1)} b_{k,m}^{1/2} b_{k+1,m}^{1/2} \leq \left( \sum_{m=p(k)}^{q(k+1)} b_{k,m} \right)^{1/2} \left( \sum_{m=p(k)}^{q(k+1)} b_{k+1,m} \right)^{1/2} \leq 1.$$

So (4.33) is bounded below by

$$C_1 C_2 \left( \sum_{m=p(k)}^{q(k+1)} (a_{k,m} - a_{k+1,m}) b_{k,m}^{1/2} b_{k+1,m}^{1/2} \right)^2.$$

Now let us denote

$$R_k = \sum_{m=p(k)}^{q(k+1)} (a_{k,m} - a_{k+1,m}) b_{k,m}^{1/2} b_{k+1,m}^{1/2}.$$

We have  $R_k = S_k + T_k + U_k + V_k$ , where

$$S_k = \sum_{m=p(k)}^{q(k)} a_{k,m} b_{k,m} - \sum_{m=p(k+1)}^{q(k+1)} a_{k+1,m} b_{k+1,m} = a_k - a_{k+1},$$

$$T_k = \sum_{m=p(k)}^{q(k+1)} a_{k,m} b_{k,m}^{1/2} (b_{k+1,m}^{1/2} - b_{k,m}^{1/2}),$$

$$U_k = \sum_{m=p(k)}^{q(k+1)} -a_{k+1,m} b_{k+1,m}^{1/2} (b_{k,m}^{1/2} - b_{k+1,m}^{1/2}),$$

$$V_k = -\delta_{k,r(i)} a_{k,q(k)} b_{k,q(k)} + \delta_{k,i-r(i)} a_{k+1,p(k+1)} b_{k+1,p(k+1)},$$

where  $\delta_{x,y} = 1$  when  $x = y$  and  $\delta_{x,y} = 0$  when  $x \neq y$ . In other words  $V_k = 0$  when  $k \neq r(i)$  and  $k \neq i - r(i)$ . In order to see why an extra term  $V_k$  appears let us recall the definition of  $p(k)$  and  $q(k)$ . We have  $q(k) = \lfloor N/i \rfloor$  for  $k = 1, \dots, r(i)$ ,  $q(k) = \lfloor N/i \rfloor - 1$  for  $k = r(i) + 1, \dots, i$  and  $p(k) = -\lfloor N/i \rfloor$  for  $k = 1, \dots, i - r(i)$ ,  $p(k) = -\lfloor N/i \rfloor - 1$  for  $k = i - r(i) + 1, \dots, i$ . A nontrivial term  $V_k$  appears only if  $q(k) \neq q(k+1)$  ( $k = r(i)$ ) or  $p(k) \neq p(k+1)$  ( $k = i - r(i)$ ).

We know that

$$(a + b + c + d)^2 \geq (a^2/4) - b^2 - c^2 - d^2, \quad a, b, c, d \in R,$$

so

$$R_k^2 = (S_k + T_k + U_k + V_k)^2 \geq (S_k^2/4) - T_k^2 - U_k^2 - V_k^2.$$

By (4.32) we obtain

$$\sum_{k=1}^{i-1} S_k^2 = \sum_{k=1}^{i-1} (a_{k+1} - a_k)^2 \geq \frac{1}{64i^3}.$$

We have already obtained that

$$\begin{aligned}
2\mathcal{A}_{2,-1}^{-1}(\lambda_2 - \lambda_1) &\geq 2C_1C_2 \sum_{k=1}^{i-1} R_k^2 \\
&\geq 2C_1C_2 \left( \frac{1}{4} \sum_{k=1}^{i-1} S_k^2 - \sum_{k=1}^{i-1} (T_k^2 + U_k^2 + V_k^2) \right) \\
&\geq 2C_1C_2 \left( \frac{1}{256i^3} - \sum_{k=1}^{i-1} (T_k^2 + U_k^2 + V_k^2) \right). \quad (4.34)
\end{aligned}$$

Now we have to estimate  $\sum_{k=1}^{i-1} (T_k^2 + U_k^2 + V_k^2)$ .

By Schwarz inequality we obtain

$$T_k^2 \leq \sum_{m=p(k)}^{q(k+1)} a_{k,m}^2 b_{k,m} |b_{k+1,m}^{1/2} - b_{k,m}^{1/2}| \sum_{m=p(k)}^{q(k+1)} |b_{k+1,m}^{1/2} - b_{k,m}^{1/2}|. \quad (4.35)$$

We have  $b_{k,m} = \int_{D_{k,m}} \varphi_1^2 = \int_{D_{k+mi}} \varphi_1^2$ . The sequence  $\left\{ \left( \int_{D_l} \varphi_1^2 \right)^{1/2} \right\}_{l=-N+1}^{l=N}$  is unimodal and its maximum is equal to  $\left( \int_{D_1} \varphi_1^2 \right)^{1/2}$ .

Now there is a very important observation in the proof of this lemma. By the unimodality of this sequence we have

$$\sum_{m=p(k)}^{q(k+1)} |b_{k+1,m}^{1/2} - b_{k,m}^{1/2}| \leq 2 \left( \int_{D_1} \varphi_1^2 \right)^{1/2}.$$

We also have  $|b_{k+1,m}^{1/2} - b_{k,m}^{1/2}| \leq \left( \int_{D_1} \varphi_1^2 \right)^{1/2}$ .

On the other hand by Lemma 4.1 we know that  $\|\varphi_1^2\|_\infty \leq 9/L$  and the area  $|D_1| = 2L/N$ . Hence  $\int_{D_1} \varphi_1^2 \leq 18/N$ .

By (4.35) we obtain

$$T_k^2 \leq \frac{36}{N} \sum_{m=p(k)}^{q(k+1)} a_{k,m}^2 b_{k,m}.$$

Similarly we get

$$U_k^2 \leq \frac{36}{N} \sum_{m=p(k)}^{q(k+1)} a_{k+1,m}^2 b_{k+1,m}.$$

Now we estimate  $V_k^2$ . Recall that  $b_{k,m} \leq \int_{D_1} \varphi_1^2 \leq 18/N$ . We have

$$\begin{aligned} V_k^2 &\leq 2(\delta_{k,r(i)} a_{k,q(k)}^2 b_{k,q(k)}^2 + \delta_{k,i-r(i)} a_{k+1,p(k+1)}^2 b_{k+1,p(k+1)}^2) \\ &\leq (36/N)(\delta_{k,r(i)} a_{k,q(k)}^2 b_{k,q(k)} + \delta_{k,i-r(i)} a_{k+1,p(k+1)}^2 b_{k+1,p(k+1)}) \end{aligned}$$

It follows that  $\sum_{k=1}^{i-1} (T_k^2 + U_k^2 + V_k^2)$  is bounded above by

$$\frac{36}{N} \sum_{k=1}^{i-1} \sum_{m=p(k)}^{q(k)} a_{k,m}^2 b_{k,m} + \frac{36}{N} \sum_{k=1}^{i-1} \sum_{m=p(k+1)}^{q(k+1)} a_{k+1,m}^2 b_{k+1,m}.$$

Note also that  $a_{k,m}^2 b_{k,m} \leq \int_{D_{k,m}} f^2 \varphi_1^2$ . Hence

$$\sum_{k=1}^{i-1} (T_k^2 + U_k^2 + V_k^2) \leq 2 \frac{36}{N} \int_D f^2 \varphi_1^2 = \frac{72}{N}.$$

This and (4.34) gives the assertion of the lemma.  $\square$

## 5 Spectral gap for convex double symmetric domains

**Proposition 5.1.** *Let  $D \subset [-L, L] \times [-1, 1]$  be open, convex and symmetric with respect to both axis. Assume  $(L, 0) \in \overline{D}$ ,  $(0, 1/2) \in \overline{D}$  and  $L = l^2 + 4$  for some natural number  $l \geq 3$ . Then*

$$\int_D f^2(x) \varphi_1^2(x) dx \leq 2 \cdot 10^9 c_H L^2 \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{2+\alpha}} \varphi_1(x) \varphi_1(y) dx dy$$

for all  $f \in L^2(D)$  such that  $\int_D f(x) \varphi_1^2(x) dx = 0$ , where  $c_H$  denotes the constant from Corollary 3.7.

*Proof.* We denote by  $D(a, b)$  the set  $D \cap ((a, b) \times \mathbf{R})$ , or  $D \cap ((a, b] \times \mathbf{R})$ , or  $D \cap ([a, b) \times \mathbf{R})$ . The latter three sets differ only by a set of a measure zero, thus the ambiguity of the definition of  $D(a, b)$  will be irrelevant. We put  $w(D(a, b)) = b - a$ , which is the “width” of  $D(a, b)$ , and

$$\begin{aligned} h(D(a, b)) &= 2 \inf\{t : (x, t) \in D(a, b) \text{ for some } x \in (a, b)\}, \\ H(D(a, b)) &= 2 \sup\{t : (x, t) \in D(a, b) \text{ for some } x \in (a, b)\}, \end{aligned}$$

the “heights” of the set  $D(a, b)$ .

Let  $\mu = \varphi_1^2 dx$ . We fix an arbitrary  $f \in L^2(D, \mu)$  such that  $\int_D f d\mu = 0$  and put  $F(x, y) = \frac{(f(x)-f(y))^2}{|x-y|^{2+\alpha}} \varphi_1(x)\varphi_1(y)$ .

Step 1. We consider a partition of  $D$  into a union of five disjoint sets  $D_1 = D(-L, -L+4)$ ,  $D_2 = D(-L+4, -L+8)$ ,  $D_3 = D(-L+8, L-8)$ ,  $D_4 = D(L-8, L-4)$  and  $D_5 = D(L-4, L)$ . Note that by unimodality and symmetry of  $\varphi_1^2$  and  $w(D_k)$ , the sequence  $\mu_k = \int_{D_k} \varphi_1^2 d\mu$  is also unimodal. Thus by Lemma 4.3 we have

$$\begin{aligned}
& 2 \int_D f^2 d\mu \leq 2 \sum_{k=1}^5 \frac{1}{\mu_k} \int_{D_k} \int_{D_k} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\
& + 100 \sum_{k=1}^4 \frac{1}{\mu_k \vee \mu_{k+1}} \int_{D_k} \int_{D_{k+1}} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\
& \leq 2(20)^{1+\alpha/2} c_3 L^2 \sum_{k=1, k \neq 3}^5 \int_{D_k} \int_{D_k} F(x, y) dx dy \\
& + 100(68)^{1+\alpha/2} 2c_3 L^2 \left( \int_{D_1} \int_{D_2} + \int_{D_4} \int_{D_5} \right) F(x, y) dx dy \\
& + \frac{100}{\mu_2 + \mu_3 + \mu_4} \iint_{(D_2 \cup D_3 \cup D_4)^2} (f(x) - f(y))^2 \mu(dx) \mu(dy). \quad (5.1)
\end{aligned}$$

In the above inequality we have used the fact that  $\sup_E \varphi_1^2 \leq c_3 L^2 \int_E \varphi_1^2 dx$  for  $E = D_k$ , where  $k \neq 3$ , or  $E = D_1 \cup D_2$  or  $E = D_4 \cup D_5$ . It turns out that one may take  $c_3 = 9c_H$  (This follows from Corollary 3.7 an argument of a geometric nature is omitted). Moreover,  $\text{diam}(D_k) \leq \sqrt{20}$  for  $k \neq 3$  and  $\text{diam}(D_k \cup D_{k+1}) \leq \sqrt{68}$  for  $k = 1, 4$ .

In this step we have “cut off” the ends of  $D$ , in a sense that it remains to estimate from above the term in (5.1)

Step 2. We now define a sequence  $a_k = l^2 - \sum_{j=1}^k (2j-1)$  for  $k = 1, 2, \dots, l$ , and  $a_0 = l^2$ . Note that  $a_l = 0$ . We consider a partition of  $D(-l^2, l^2) = D_2 \cup D_3 \cup D_4$  into a union of  $2l$  pairwise disjoint sets  $D'_{-k} = D(-a_{k-1}, -a_k)$  and  $D'_k = D(a_k, a_{k-1})$  for  $k = 1, 2, \dots, l$ . Let  $\mu'_k = \mu(D'_k)$ . By a similar token as before, the sequence  $(\mu'_{-1}, \mu'_{-2}, \dots, \mu'_{-l}, \mu'_l, \mu'_{l-1}, \dots, \mu'_1)$  is unimodal. Thus

by Lemma 4.3 and the equality  $\mu'_k = \mu'_{-k}$  we have

$$\begin{aligned} & \frac{1}{\mu(D(-l^2, l^2))} \iint_{D(-l^2, l^2)^2} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\ \leq & 2 \sum_{k=1}^l \frac{1}{\mu'_k} \left( \int_{D'_k} \int_{D'_k} + \int_{D'_{-k}} \int_{D'_{-k}} \right) (f(x) - f(y))^2 \mu(dx) \mu(dy) \quad (5.2) \end{aligned}$$

$$+ 16l^2 \sum_{k=1}^{l-1} \frac{1}{\mu'_{k+1} \vee \mu'_k} \left( \int_{D'_{-k}} \int_{D'_{-k-1}} + \int_{D'_{k+1}} \int_{D'_k} \right) (f(x) - f(y))^2 \mu(dx) \mu(dy) \quad (5.3)$$

$$+ 16l^2 \frac{1}{\mu'_l} \int_{D'_{-l}} \int_{D'_l} (f(x) - f(y))^2 \mu(dx) \mu(dy). \quad (5.4)$$

Step 3. We will now show how to deal with the integral (5.3), i.e.,

$$\begin{aligned} I_k &= \frac{1}{\mu'_{k+1} \vee \mu'_k} \int_{D'_{k+1}} \int_{D'_k} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\ &\leq \frac{1}{\mu'_{k+1} + \mu'_k} \iint_{(D'_{k+1} \cup D'_k)^2} (f(x) - f(y))^2 \mu(dx) \mu(dy). \quad (5.5) \end{aligned}$$

We have  $w = w(D'_{k+1} \cup D'_k) = a_{k-1} - a_{k+1} = 4k$ . Let  $h = h(D'_{k+1} \cup D'_k)$  and  $N = \lceil \frac{2k+1}{h} \rceil$ . We divide  $D'_{k+1} \cup D'_k$  into a union of sets

$$E_j = D(a_{k+1} + (j-1)w/N, a_{k+1} + jw/N), \quad j = 1, 2, \dots, N,$$

of equal width  $4k/N$  and apply Lemma 4.3 to such  $E_j$ . We obtain

$$\begin{aligned} I_k &\leq 4 \sum_{j=1}^N \frac{1}{\mu(E_j)} \int_{E_j} \int_{E_j} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\ &\quad + 8N^2 \sum_{j=1}^{N-1} \frac{1}{\mu(E_j) \vee \mu(E_{j+1})} \int_{E_j} \int_{E_{j+1}} (f(x) - f(y))^2 \mu(dx) \mu(dy). \end{aligned}$$

Note that

$$\text{dist}((D'_{k+1} \cup D'_k), (L, 0)) = 4 + (k-1)^2 \geq 2k = w(D'_{k+1} \cup D'_k)/2,$$

thus by convexity of  $D$  we have

$$\frac{H(D'_{k+1} \cup D'_k)}{h(D'_{k+1} \cup D'_k)} \leq \frac{\text{dist}((D'_{k+1} \cup D'_k), (L, 0)) + w(D'_{k+1} \cup D'_k)}{\text{dist}((D'_{k+1} \cup D'_k), (L, 0))} \leq 3.$$

Hence  $h \leq H(E_j) \leq 3h$ . Moreover,

$$w(E_j \cup E_{j+1}) = \frac{8k}{N} \leq 8h \frac{k}{2k+1-h} \leq 4h$$

and

$$w(E_j \cup E_{j+1}) = \frac{8k}{N} \geq h \frac{8k}{2k+1} \geq \frac{8}{3}h,$$

This means that if  $S = a_{k+1} + (j-1)w/N$ , then  $B((S, 0), h) \subset D$  and  $|B((S, 0), h/4) \cap E_j| = |B((S, 0), h/4)|/2 = \pi h^2/32$ . Thus by Harnack inequality (Corollary 3.7) we obtain

$$\sup_{E_j} \varphi_1^2 \leq \frac{32c_H}{\pi h^2} \int_{E_j} \varphi_1^2 dx,$$

the same bound as above holds also for  $E_j \cup E_{j+1}$  in place of  $E_j$ . Note that we take  $h/4$  above as the radius of the ball because of the assumption concerning inner radius in Corollary 3.7.

We have

$$\text{diam}(E_j) \leq \text{diam}(E_j \cup E_{j+1}) \leq (w(E_j \cup E_{j+1})^2 + H(E_j \cup E_{j+1})^2)^{1/2} \leq 5h.$$

Hence

$$\begin{aligned} & \frac{1}{\mu(E_j) \vee \mu(E_{j+1})} \int_{E_j} \int_{E_{j+1}} (f(x) - f(y))^2 \mu(dx) \mu(dy) \\ & \leq (5h)^{2+\alpha} \frac{32c_H}{\pi h^2} \int_{E_j} \int_{E_{j+1}} F(x, y) dx dy \end{aligned}$$

and a similar bound holds for the integral over  $E_j \times E_j$ . Moreover,  $N^2 \leq (2k+1)^2/h^2 \leq 9k^2/h^2$ . Thus

$$\begin{aligned} I_k & \leq 4(5h)^{2+\alpha} \frac{32c_H}{\pi h^2} \sum_{j=1}^N \int_{E_j} \int_{E_j} F(x, y) dx dy \\ & \quad + \frac{72k^2}{h^2} \cdot (5h)^{2+\alpha} \frac{32c_H}{\pi h^2} \sum_{j=1}^{N-1} \int_{E_j} \int_{E_{j+1}} F(x, y) dx dy \\ & \leq \frac{2304 \cdot 5^{2+\alpha} c_H}{\pi} \cdot k^2 h^{\alpha-2} \iint_{(D'_{k+1} \cup D'_k)^2} F(x, y) dx dy. \end{aligned} \quad (5.6)$$

We have  $\text{dist}((D'_{k+1} \cup D'_k), (L, 0)) = 4 + (k-1)^2 \geq k^2/2$ , thus that by convexity of the set  $D$  and the assumptions  $(0, 1/2) \in \overline{D}$ ,  $(L, 0) \in \overline{D}$  we obtain

$$\frac{h/2}{k^2/2} \geq \frac{1/2}{L}.$$

When  $\alpha \geq 1$  we get  $k^2 h^{\alpha-2} \leq 2L h h^{\alpha-2} \leq 2L$ . When  $\alpha < 1$  we get  $k^2 h^{\alpha-2} \leq k^2 (2L/k^2)^{2-\alpha} = 2^{2-\alpha} L (L/k^2)^{1-\alpha} \leq 2^{2-\alpha} L$ . For any  $\alpha \in (0, 2)$  we have  $k^2 h^{\alpha-2} \leq \max(2, 2^{2-\alpha}) L$ . We combine it with (5.6) and finally obtain

$$I_k \leq \frac{2880000 c_H}{\pi} \cdot L \iint_{(D'_{k+1} \cup D'_k)^2} F(x, y) dx dy. \quad (5.7)$$

We should also estimate from above the integral (5.4) over  $D_{-l} \times D_l$ . This may be done in a similar way as the integrals  $I_k$  above. We obtain a similar estimate as (5.7) with slightly smaller constant, we omit the details.

To estimate (5.2) we may see that in (5.5) we have in fact estimated  $I_k$  from above by an integral over  $(D'_{k+1} \cup D'_k)^2$ . Thus a similar estimation as in (5.7) holds also for the integrals in (5.2).

We finally obtain

$$\begin{aligned} 2 \int_D f^2 d\mu &\leq \left( 18 \cdot (20)^{1+\alpha/2} + 1800 \cdot (68)^{1+\alpha/2} + 100 \left( \frac{4}{L} + 32 \right) \cdot \frac{2880000}{\pi} \right) \\ &\times c_H L^2 \iint_{D^2} F(x, y) dx dy \end{aligned}$$

and the proposition follows.  $\square$

*Proof of Theorem 1.4.* By scaling of eigenvalues (1.6) it is sufficient to consider domains  $D$  such that  $[-L, L] \times [-1, 1]$ ,  $L \geq 1$  is the smallest rectangle (with sides parallel to the coordinate axes) containing  $D$  and to show that for such domains the following inequality holds:

$$2\mathcal{A}_{2,-\alpha}^{-1}(\lambda_2 - \lambda_1) \geq \frac{C}{L^2},$$

where  $C$  is the same as in (1.9).

At first assume that  $L \geq 13$ . For any natural number  $l \geq 3$  we have

$$\frac{(l+1)^2 + 4}{l^2 + 4} \leq \frac{20}{13},$$

thus there exists  $\beta \in [13/20, 1]$  such that  $\beta L = l^2 + 4$  for some natural number  $l \geq 3$ . By (1.6) we have

$$2\mathcal{A}_{2,\alpha}^{-1}(\lambda_2(D) - \lambda_1(D)) = \beta^\alpha \cdot 2\mathcal{A}_{2,\alpha}^{-1}(\lambda_2(\beta D) - \lambda_1(\beta D)).$$

Note that  $\beta D$  satisfies assumptions of Proposition 5.1 (in particular  $(0, 1/2) \in \beta D$ ). Hence by Theorem 1.1 and Proposition 5.1 we obtain

$$\begin{aligned} 2\mathcal{A}_{2,\alpha}^{-1}(\lambda_2(D) - \lambda_1(D)) &= 2\mathcal{A}_{2,\alpha}^{-1} \beta^\alpha (\lambda_2(\beta D) - \lambda_1(\beta D)) \\ &\geq \frac{\beta^\alpha}{10^9 c_H (\beta L)^2} \geq \frac{1}{10^9 c_H L^2}. \end{aligned}$$

What remains is to consider the case  $L \leq 13$ .

Note that  $B((0,0), 1/\sqrt{2}) \subset D$  so by Corollary 3.7  $\varphi_1^2$  satisfies Harnack inequality on  $B = B((0,0), 1/(4\sqrt{2}))$ , in particular  $\varphi_1^2(0,0) \leq c_H \varphi_1^2(x)$ ,  $x \in B$ . Of course  $\sup_{x \in D} \varphi_1^2(x) = \varphi_1^2(0,0)$ . We have

$$c_H = c_H \int_D \varphi_1^2 \geq c_H \int_B \varphi_1^2 \geq \varphi_1^2(0,0)|B|,$$

which gives  $\sup_{x \in D} \varphi_1^2(x) \leq c_H |B|^{-1} = 32c_H/\pi$ .

We also have  $\text{diam}(D) < 28$ . Let  $f = \varphi_2/\varphi_1$ . By Theorem 1.1 we have

$$\begin{aligned} 2\mathcal{A}_{2,\alpha}^{-1}(\lambda_2 - \lambda_1) &= \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{2+\alpha}} \varphi_1(x) \varphi_1(y) dx dy \\ &\geq \frac{\pi}{32c_H 28^{2+\alpha}} \int_D \int_D (f(x) - f(y))^2 \varphi_1^2(x) \varphi_1^2(y) dx dy \\ &= \frac{\pi}{32c_H 28^{2+\alpha}} 2 \int_D f^2(x) \varphi_1^2(x) dx = \frac{\pi}{16c_H 28^{2+\alpha}}. \end{aligned}$$

□

**Acknowledgments.** The second name author thanks very much Rodrigo Bañuelos for many interesting discussions on the subject of the paper.

## References

- [1] R. Bañuelos, *Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators*, J. Funct. Anal. 100 (1991), 181-206.
- [2] R. Bañuelos, T. Kulczycki, *The Cauchy process and the Steklov problem*, J. Funct. Anal. 211(2) (2004), 355-423.
- [3] R. Bañuelos, T. Kulczycki, *Eigenvalue gaps for the Cauchy process and a Poincare inequality*, J. Funct. Anal. 234(1) (2006) 199-225.
- [4] R. Bañuelos, T. Kulczycki, *Spectral gap for the Cauchy process on convex, symmetric domains*, Comm. Partial Differential Equations 31(12)(2006), to appear.
- [5] R. Bañuelos, T. Kulczycki, P. J. Méndez-Hernández, *On the shape of the ground state eigenfunction for stable processes*, Potential Anal. 24(3) (2006), 205-221.

- [6] R. Bañuelos, R. Latała, P. J. Méndez-Hernández, *A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes*, Proc. Amer. Math. Soc. 129(10) (2001), 2997–3008 (electronic).
- [7] R. Bañuelos, P. J. Méndez-Hernández, *Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps*, J. Funct. Anal. 176(2) (2000), 368–399.
- [8] R.M. Blumenthal and R.K. Gettoor, *The asymptotic distribution of the eigenvalues for a class of Markov operators* Pacific J. Math. 9 (1959), 399–408.
- [9] R.M. Blumenthal, R.K. Gettoor, D.B. Ray *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. 99 (1961) 540–554.
- [10] K. Bogdan and T. Byczkowski *Potential theory for the  $\alpha$ -stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. 133(1) (1999), 53–92.
- [11] K. Burdzy, T. Kulczycki *Stable processes have thorns*, Ann. Probab. 31(1) (2003), 170–194.
- [12] Z.Q. Chen and R. Song, *Intrinsic ultracontractivity and conditional gauge for symmetric stable processes*, J. Funct. Anal. 150(1) (1997), 204–239.
- [13] Z.Q. Chen and R. Song, *Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains*, Illinois J. Math. 44(1) (2000), 138–160.
- [14] Z.Q. Chen and R. Song, *Two sided eigenvalue estimates for subordinate Brownian motion in bounded domains*, J. Funct. Anal. 226 (2005), 90–113.
- [15] Z.Q. Chen and R. Song, *Continuity of eigenvalues for subordinate processes in domains*, Math. Z. (2005), (to appear).
- [16] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.
- [17] B. Davis, *On the spectral gap for fixed membranes*, Ark. Mat. 39(1) (2001), 65–74.

- [18] R. D. DeBlassie, *Higher order PDEs and symmetric stable processes* Probab. Theory Related Fields 129 (2004), 495–536.
- [19] R. D. DeBlassie and P. J. Méndez-Hernández,  *$\alpha$ -continuity properties of symmetric  $\alpha$ -stable process*, Preprint.
- [20] R. K. Gettoor, *First passage times for symmetric stable processes in space*, Trans. Amer. Math. Soc. 101 (1961) 75–90.
- [21] R. K. Gettoor, *Markov operators and their associated semi-groups*, Pacific J. Math. 9 (1959) 449–472.
- [22] N. Ikeda, S. Watanabe *On some relations between the harmonic measure and the Levy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. 2 (1962), 79–95.
- [23] T. Kulczycki, *Intrinsic ultracontractivity for symmetric stable processes*, Bull. Polish Acad. Sci. Math. 46(3) (1998), 325–334.
- [24] J. Ling, *A lower bound for the gap between the first two eigenvalues of Schrödinger operators on convex domains in  $S^n$  or  $R^n$* , Michigan Math. J. 40(2) (1993), 259–270.
- [25] P. J. Méndez-Hernández, *Brascamp-Lieb-Luttinger Inequalities for Convex Domains of Finite Inradius*, Duke Math. Journal 113 (2002), 93–131.
- [26] I. M. Singer, B. Wong, S.-T. Yau, S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12(2) (1985), 319–333.
- [27] R. G. Smits, *Spectral gaps and rates to equilibrium for diffusions in convex domains* Michigan Math. J. 43(1) (1996), 141–157.
- [28] Q. Yu and J. Q. Zhong, *Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator*, Trans. Amer. Math. Soc. 294 (1986), 341–349.
- [29] V. M. Zolotarev, *Integral transformations of distributions and estimates of parameters of multidimensional spherically symmetric stable laws*, in: Contributions to Probability, Academic Press, New York, (1981), 283–305.